

# The Economics of Hedge Funds: Alpha, Fees, Leverage, and Valuation\*

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## Abstract

Hedge fund managers charge management fees on assets under management (AUM) and incentive fees indexed to the high-water mark (HWM). We study the effects of fees and alpha on managerial dynamic leverage choice and valuation. Our main results are: (i) high-powered incentive fees encourage excessive risk taking, while management fees have the opposite effect; (ii) agency conflicts have significant effects on dynamic leverage choices and valuation of managerial rents and investors' payoffs; (iii) the manager's optimal leverage critically depends on the history of the fund's performance; (iv) investors' options to withdraw and/or liquidate funds following sufficiently poor fund performance substantially curtail managerial risk-taking and can give rise to strong precautionary cash holding; and (v) managerial ownership concentration has strong incentive alignment effects.

**Keywords:** high-water mark, management fees, incentive fees, funding costs, conflicts of interest, agency, managerial ownership, liquidation option, AUM

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# 1 Introduction

Hedge funds' management compensation contracts typically feature both management fees and performance/incentive fees. The management fee is charged periodically as a fraction, e.g. 2%, of assets under management (AUM). The incentive fee, a key characteristic that differentiates hedge funds from mutual funds, is calculated as a fraction, e.g. 20%, of the fund's profits. The cost base for the profit calculation is the fund's high-water mark (HWM), which effectively keeps track of the maximum value of the invested capital and critically depends on the fund manager's dynamic investment strategies. Presumably, incentive fees are intended to reward talented managers and to align the interests of the manager and investors more closely than flat management fees do. However, incentive fees may also have unintended consequences because they tend to encourage managerial excessive risk taking and do not lead to fund value maximizing leverage choices.<sup>1</sup>

One important feature of hedge funds is their sophisticated use of leverage (Ang, Gorovyy, and van Inwegen (2010)). Hedge funds may borrow through the repo markets or from their prime brokers. Hedge funds can also use various implicit leverage, often via options and other derivatives. Skilled managers, by leveraging on strategies with positive alphas, can potentially create significant value for investors. That is, with a limited amount of capital, leverage can be rewarding for both skilled managers and the funds' investors. However, there are various funding costs associated with taking on leverage or other financing. For example, the haircut on collateral assets and the margin requirements for borrowing make external financing costly. Moreover, the bigger the financing, the higher the marginal financing cost is. The manager takes into account both the funding cost and the alpha generating technology when choosing dynamic investment strategies.

We develop an analytically tractable framework of leverage and valuation for hedge funds with the following features: (i) an alpha generating technology, (ii) the management fee specified as a fraction of AUM; (iii) the high-powered incentive fee linked to the HWM, and (iv) the cost of borrowing funds for leverage (haircuts/margins). The manager dynamically chooses leverage to maximize his total managerial rents, the sum of present values of both

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<sup>1</sup>Private equity funds also charge both management and incentive fees. While the compensation structure is similar in essence, institutional details such as fees and profits calculations differ significantly for hedge funds and private equity funds. Metrick and Yasuda (2010) provide an economic analysis of private equity funds.

management and incentive fees.

For both conceptual and quantitative reasons, it is important to incorporate both types of fees in our analysis. We will show that management and incentive fees have different and often opposite implications on managerial risk taking. With management fees only, the manager effectively behaves in the investors' interest because management fees make the manager a *de facto* equity investor in the fund and there is no agency issue between the manager and investors.<sup>2</sup> However, when the manager is paid via both fees, conflicts of interest between the manager and investors arise. The optionality embedded in incentive fees encourage the manager to choose leverage more than the desired level from the investors' perspective. The closer the fund's AUM is to its HWM, the more leverage the manager chooses because the incentive fee is closer to being realized (i.e. the embedded call option is deeper in the money). This standard optionality argument suggests that the present value (PV) of incentive fees is increasing and *convex* in AUM.

On the other hand, paying incentive fees lower the AUM and hence reduce the future management fees. This creates a tension and an opposing effect between the PV of management fees and the PV of incentive fees. Importantly, the PV of management fees is increasing but *concave* in AUM. As a result, management fees encourage the manager to lower leverage. However, the effect of incentive fees dominates that of management fees, especially when the fund does well (i.e. close to its HWM) because the manager is the sole recipient of incentive fees. As a result, the manager effectively behaves in an risk-seeking way in the absence of other frictions. In addition to deriving the manager's dynamic leverage decisions, we also explicitly link the PV of managerial compensation to managerial talent/alpha. Our model thus provides a valuation toolbox for investors to evaluate different managerial compensation contracts and their investments given their beliefs about managerial talent (e.g. the unlevered alpha and the Sharpe ratio).

We extend our model to allow investors to liquidate the fund when the fund's performance is sufficiently poor. The investors' liquidation/withdrawal option allows investors to better manage their downside risk exposure. The fund manager is short in the liquidation option, and rationally manages the fund's leverage in order to maximize the PV of total fees. Quantitatively, we show that liquidation options significantly curtail managerial risk-taking and

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<sup>2</sup>See Duffie, Wang, and Wang (2009) for a similar result in a different setting with management fees only.

sometimes cause the manager to be too conservative from the perspective of investors. We also find that concentrated managerial ownership helps to align the incentives between the manager and investors. Our model provides the first elements of an analytically tractable operational framework to study the effects of managerial compensation on managerial leverage and dynamic valuation for the managerial and investors' stakes.

**Related literature.** There are only a few papers studying the hedge fund's valuation and leverage decisions. Goetzman, Ingersoll, and Ross (2003), henceforth abbreviated GIR, is the first to provide an intertemporal valuation framework for management and incentive fees. One key ingredient of their paper is the high-water mark. They take AUM dynamics as given and do not allow for managerial leverage decisions. Building on this valuation framework, we introduce funding costs and derive time-varying optimal leverage and implied valuation.

Panageas and Westerfield (2009) study the manager's investment decisions when the manager receives incentive fees without management fees. They obtain explicit time-invariant investment strategies, similar to portfolio choice problems for investors with constant relative risk aversion as in Merton (1971). While the solution may be similar, the economics is rather different. In terms of the model setup, our paper differs from theirs by incorporating management fees and also the funding costs of leverage. We show that optimal leverage chosen by the manager critically depends on the ratio of AUM and the high-water mark due to the interaction between both incentive and management fees. Management and incentive fees play important but different roles on leverage and valuation. In terms of results, the value of incentive fees in our paper is convex in AUM, while it is concave in their model with no liquidation boundary.<sup>3</sup>

Duffie, Wang, and Wang (2009) study leverage management when the manager is compensated via management fees but not incentive fees. In the absence of high-powered incentives, the manager is effectively a co-investor in the fund whose cash flows are management fees proportional to AUM, and thus effectively behaves in the investors' interest by optimally balancing the trading costs (e.g. bid/ask spread) with the benefits of leverage on alpha. Unlike their paper, our paper focuses on agency issues induced by incentive fees being indexed to the high-water mark. Dai and Sundaresan (2010) point out that the hedge fund is short

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<sup>3</sup>Other differences between the two papers include payouts to investors and the allowance for a lower liquidation boundary.

in two important options: investors' redemption option and funding options (from prime brokers and short-term debt markets). These two short positions have significant effects on optimal leverage and risk management policies (e.g. the use of unencumbered cash). Like their paper, we also model the investors' liquidation option. However, we do not model the time-varying nature of the external funding options. They do not model the effects of incentive fees on leverage and valuation. Additionally, our paper also studies the importance of managerial ownership on mitigating agency conflicts.

There has been much recent and continuing interest in empirical research on hedge funds. Fung and Hsieh (1997), Ackermann, McEnally, and Ravenscraft (1999), Agarwal and Naik (2004), and Getmansky, Lo, and Makarov (2004) among others, study the nonlinear feature of hedge fund risk and return.<sup>4</sup> Lo (2008) provides a detailed treatment of hedge funds for their potential contribution to systemic risk in the economy.

## 2 The baseline model

First, we introduce the fund's investment opportunity and its funding cost function. Second, we describe the fund's managerial compensation contracts, including both management and incentive fees. Then, we discuss the dynamics for the fund's AUM, and define various value functions for the manager and investors. Finally, we state the manager's intertemporal optimization problem subject to investors' voluntary participation.

**The fund's investment technology.** The fund has a trading strategy which generates expected excess returns after risk adjustments. The fund's expected excess returns may be attributed to managerial talent and may not be traded. In reality, fund managers are often secretive about their trading strategies and sometimes take measures to make replications of their strategies difficult.

Let  $dR(t)$  denote the instantaneous return for the fund's trading strategy with no leverage. This strategy generates independently and identically distributed (iid) returns as follows:

$$dR(t) = \mu dt + \sigma dB(t), \tag{1}$$

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<sup>4</sup>For the presence of survivorship bias, selection bias, and back-filling bias in hedge funds databases, see Brown, Goetzmann, Ibbotson, and Ross (1992) for example.

where  $B(t)$  is a standard Brownian motion. The parameters  $\mu$  gives the expected return after systematic risk adjustments such as the market risk, and the parameter  $\sigma$  is the volatility for the strategy, respectively. The risk-free rate  $r$  is a constant. We define  $\alpha$ , the expected excess return as follows:

$$\alpha = \mu - r. \tag{2}$$

Positive alpha measures managerial talent. Scarce managerial skills earn rents in equilibrium as we will show.

The fund can invest its assets under management (AUM) in both the alpha-generating strategy defined by (1)-(2) and the risk-free asset which pays constant rate of interest  $r$ . Let  $W$  denote the fund's AUM. The manager may find it optimal to leverage on its alpha-generating technology in order to increase managerial rents. Higher leverage delivers a larger expected excess return but also results in higher volatility. Moreover, leverage induces funding and other costs. Naturally, the fund's investment strategy depends on the funding costs of its overall position. We next turn to the specification of the funding costs.

**The funding cost.** Let the amount of investment in the risk-free asset be  $D$ . With AUM (i.e. capital)  $W$ , the amount invested in the risky alpha-generating technology is then  $W - D$ . When  $D < 0$ , the fund takes on leverage, often via short-term debt. A common way for hedge funds to obtain leverage is from the fund's prime brokers.<sup>5</sup> To model financing frictions in a parsimonious way, we assume financing is costly as in standard capital structure models in corporate finance. For example, lenders protect themselves against downside risk by asking for margin requirements and haircut for collateral.

We denote the asset-capital ratio as  $\pi = (W - D)/W$ . Leverage implies  $\pi > 1$ . We assume that the funding cost  $\Gamma(\pi, W)$  takes the following homogenous form:

$$\Gamma(\pi, W) = \gamma(\pi)W. \tag{3}$$

The above functional form implies that the total funding cost  $\Gamma(\pi, W)$  doubles if we double both the amount of borrowing  $|D|$  and capital  $W$ . Note that leverage  $\pi$  remains unchanged when both  $W$  and the amount of borrowing  $|D|$  double. Intuitively, the more the fund borrows, the higher the financing cost, and moreover, the higher the *marginal* financing

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<sup>5</sup>Few hedge funds are able to directly issue long-term debt or secure long-term borrowing. Not all hedge funds use prime brokers.

cost. The financing cost function  $\gamma(\pi)$  is increasing and convex:  $\gamma'(\pi) > 0$  and  $\gamma''(\pi) \geq 0$  for  $\pi > 1$ . If the fund takes a long position in both the risky asset and the risk-free asset, then there is no financing cost, i.e.  $\gamma(\pi) = 0$  for  $0 \leq \pi \leq 1$ . As we will show later, if the liquidation risk and redemption risk are sufficiently high, the manager may take a less levered and a more conservative position.

**Managerial compensation contracts.** The typical hedge fund management compensation contract has two main components: the management fees and the incentive fees. The management fee is specified as a constant fraction  $c$  of the AUM  $W$ :  $\{cW(t) : t \geq 0\}$ . The incentive fee links compensation to the fund's performance.

To describe the incentive fees, we need to understand the fund's high-water mark process  $\{H(t) : t \geq 0\}$ . For the purpose of exposition, first consider the simplest example where the high-water mark  $H_t$  is the highest level that the AUM  $W$  has attained up to time  $t$ , i.e.  $H$  is the running maximum of  $W$ :  $H(t) = \max_{s \leq t} W(s)$ .

More generally, the high-water mark (HWM) may also change due to indexed growth or investors' withdrawal.<sup>6</sup> Let  $g$  denote the (normal) rate at which  $H$  grows. As in GIR, investors in our model are paid continuously at a rate  $\delta W(t)$  where  $\delta$  is a constant. Naturally, the fund's high-water mark is adjusted downward due to investors' withdrawal at the rate  $\delta$ .

Provided that the fund is in operation and the fund's AUM  $W$  is below its high-water mark ( $W < H$ ), the evolution of  $H$  is locally deterministic and is given by

$$dH(t) = (g - \delta)H(t)dt, \quad \text{if } W(t) < H(t). \quad (4)$$

For example,  $g$  may be set to zero, or the interest rate  $r$  or other benchmark levels. This growth of  $H$  to some extent captures the time value of money. Whenever the AUM  $W$  exceeds its running maximum, the manager collects a fraction  $k$  of the fund's performance exceeding its high-water mark and then the high-water mark resets. We provide detailed analysis of the dynamics of the high-water mark  $H$  on the boundary  $W = H$  in Section 4. We often see that  $c$  and  $k$  are around 2% and 20%, respectively. Of course, managers

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<sup>6</sup>Sometimes, the high-water mark is also negotiated downward if the fund has performed poorly. Sometimes, it is argued that resetting the high-water mark helps to re-align the incentives between the manager and the investors. To offset the potential increase in incentive fees collected, the manager in exchange may reduce management fees by lowering  $c$ .

with different track records charge differently. Our analysis allows us to link the managerial compensation contract to their skill, e.g. alpha.

**The dynamic of AUM  $W$ .** In our baseline model and in GIR, investors stochastically liquidate the fund at a constant rate  $\lambda$  per unit of time. This assumption makes the model stationary and analytically tractable. Upon liquidation at exogenous stochastic time  $\tau$ , the manager receives nothing and investors collect AUM  $W$ .

Before investors liquidate the fund ( $t < \tau$ ), the AUM  $W$  evolves as follows:

$$\begin{aligned} dW(t) = & \pi(t)W(t)(\mu dt + \sigma dB(t)) + (1 - \pi(t))rW(t)dt - \Gamma(\pi(t), W(t))dt \\ & - \delta W(t)dt - cW(t)dt - k[dH(t) - (g - \delta)H(t)dt] - dJ(t). \end{aligned} \quad (5)$$

The first and second terms in (5) describe the change of AUM  $W$  from the manager's investment strategies in its alpha-generating technology and the risk-free asset as in standard portfolio choice problem (Merton (1971)). The third term in (5) measures the funds' financing costs  $\Gamma(\pi, W)$  given its asset-capital ratio  $\pi$ . The fourth term  $-\delta W dt$  gives the continuous payout rate to investors. These four terms sum to give the change of  $W$  in the absence of fees. The fifth term represents the management fees to the manager in flow terms (e.g.  $c = 2\%$ ). The sixth term gives the incentive/performance fees (e.g.  $k = 20\%$ ). The manager collects the incentive fees if and only if  $dH(t) > (g - \delta)H(t)dt$ , which can only happen on the boundary ( $W(t) = H(t)$ ). However, after reaching the new HWM, the manager may not collect incentive fees if the fund does not make profits.

Investors are paid a constant rate  $\delta$  out of the fund's AUM. In addition, investors may also liquidate the whole fund. In the baseline model, we assume that investors liquidate the fund exogenously with probability  $\lambda \Delta t$  over time interval  $(t, t + \Delta t)$ , where  $\lambda$  is a constant. Upon liquidation, investors collect all AUM and the manager collects no more fees. The process  $J$  in the last (seventh) term is a pure jump process which describes this liquidation risk: The AUM  $W$  is set to zero with probability  $\lambda dt$  per unit of time.

**Various value functions for investors and the manager.** We next define present values for various streams of cash flows. For a given dynamic investment strategy  $\pi$ , we use  $M(W, H; \pi)$  and  $N(W, H; \pi)$  to denote the present value of the management fees and that of the incentive fees, respectively. Recall that the alpha-generating technology (1)-(2) is



already after the systematic risk adjustment. For example, when the market portfolio is the only source of systematic risk (i.e. CAPM holds for investments with no alpha), (1) should be interpreted as market-neutral excess return after factoring out the market risk premium. We may thus discount cash flows using the risk-free rate as follows:

$$M(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} cW(s) ds \right], \quad (6)$$

$$N(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} k [dH(s) - (g - \delta)H(s) ds] \right], \quad (7)$$

where the manager collects neither management nor incentive fees after stochastic liquidation time  $\tau$ . Let  $F(W, H; \pi)$  denote the present value of total fees, which is given by

$$F(W, H; \pi) = M(W, H; \pi) + N(W, H; \pi). \quad (8)$$

Similarly, we define investors' present value  $E(W, H)$  as follows:

$$E(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} \delta W(s) ds + e^{-r(\tau-t)} W(\tau) \right]. \quad (9)$$

Finally, the total value of the fund  $V(W, H)$  is given by the sum of  $F(W, H)$  and  $E(W, H)$ :

$$V(W, H; \pi) = F(W, H; \pi) + E(W, H; \pi). \quad (10)$$

**The manager's optimization problem.** The manager chooses the optimal dynamic investment policy  $\pi$  to maximize the value of total fees  $F(W, H; \pi)$  as follows:

$$\max_{\pi} F(W, H; \pi) = \max_{\pi} \{M(W, H; \pi) + N(W, H; \pi)\}, \quad (11)$$

subject to transversality conditions.

**Investors' voluntary participation condition.** Anticipating that the manager behaves in his own interest, investors rationally expect agency conflicts and demand that the present value of their payoffs is at least higher than their time-0 investment  $W(0)$  in the fund so that investors at least break even. At time 0, the fund's HWM is set at  $H(0) = W(0)$ . Given the manager's optimal dynamic investment strategy  $\pi$ , we thus require the investors' value function  $E(W(0), W(0); \pi)$  at time 0 to satisfy the following condition:

$$E(W(0), W(0); \pi) \geq W(0). \quad (12)$$

Before analyzing the general case where the manager receives both management and incentive fees, we first study the case where the manager only receives management fees.

### 3 Solution for a benchmark: management fees only

With only management fees and no incentive fees ( $k = 0$ ), we obtain closed-form solutions for the manager's investment strategy and various value functions for the manager and investors. As we will show, without high-powered incentive schemes ( $k = 0$ ), there are no conflicts of interest between the manager and investors. Using a different model setup, Duffie, Wang, and Wang (2009) also obtain this result when the manager is only paid management fees. We thus view our results as complementary and reinforcing theirs.<sup>7</sup> Using this special case with no incentive fees as a benchmark, we analyze the impact of incentive fees on the manager's investment strategy in Section 4. The following proposition summarizes the results for the special case with  $k = 0$ .

**Proposition 1** *With no incentive fees ( $k = 0$ ), the present value of the manager's total fees  $F^*(W)$  solely comes from management fees and is given by*

$$F^*(W) = M^*(W) = \frac{c}{c + \delta + \lambda - \alpha^*} W, \quad (13)$$

where  $\alpha^*$  is a constant given by the following equation:

$$\alpha^* = \alpha\pi^* - \gamma(\pi^*), \quad (14)$$

and  $\pi^* > 1$  is the unique optimal time-invariant investment strategy given by:

$$\alpha = \gamma'(\pi^*), \quad (15)$$

provided that  $\gamma'(1) < \alpha$ . Otherwise (whenever  $\gamma'(1) \geq \alpha > 0$ ),  $\pi^* = 1$ . With optimal investment strategy  $\pi^*$ , the present value of investors' payoff is given by

$$E^*(W) = \frac{\delta + \lambda}{c + \delta + \lambda - \alpha^*} W. \quad (16)$$

Summing the present values for the manager and investors, we obtain the present value of the total fund  $V^*(W)$ :

$$V^*(W) = M^*(W) + E^*(W) = \frac{c + \delta + \lambda}{c + \delta + \lambda - \alpha^*} W. \quad (17)$$

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<sup>7</sup>Duffie, Wang, and Wang (2009) use the proportional transaction cost setting as in the well known portfolio choice model of Davis and Norman (1990). Duffie, Wang, and Wang (2009) study managerial leverage when the manager is paid purely via management fees. Our results reported in the benchmark model of Section 3 have the same focus and similar results, despite different detailed settings.

Note that  $\alpha^*$  is the levered alpha net the cost of financing. For convergence, we need to ensure that levered alpha  $\alpha^*$  cannot be too high, i.e.

$$\alpha^* < c + \delta + \lambda, \quad (18)$$

where  $\alpha^*$  is given by (14)-(15). Intuitively, the above inequality ensures that investment strategy is not too good to be true (i.e. the value of the alpha-generating technology (1) even with leverage is still finite).

Without agency, some leverage is optimal (i.e.  $\pi^* > 1$ ) under the condition  $\gamma'(1) < \alpha$ . However, if  $\gamma'(1) > \alpha$ , the fund will optimally choose not to lever because the marginal cost even for infinitesimal leverage is greater than the corresponding marginal benefit, implying that the optimal time-invariant investment strategy is  $\pi^* = 1$ .

By charging no incentive fees and a constant rate  $c$  per unit of AUM  $W$  in management fees, the manager effectively becomes an equity investor in the fund without explicit legal ownership. Investors receive payouts at a continuous constant rate  $\delta$  per unit of AUM  $W$  over time, and completely liquidate the AUM  $W$  with constant probability  $\lambda$  per unit of time. The (continuous) payout rate to investors is thus  $(\delta + \lambda)$  per unit of AUM  $W$ . Taking the ratio between this payout rate to investors,  $\delta + \lambda$ , and the total payout rate to both the manager and investors,  $(\delta + \lambda + c)$ , we obtain the “effective” equity stake in the fund for investors:  $(\delta + \lambda)/(\delta + \lambda + c)$ . The remaining  $c/(\delta + \lambda + c)$  fraction is the manager’s effective equity stake in the fund. There is a perfect alignment of incentives between the manager and investors. The manager takes the time-invariant investment strategy  $\pi^*$  given in (15) which maximizes both the manager’s value  $F(W, H)$  and also investors’ value  $E(W, H)$ .

In our numerical analysis, we use the following quadratic financing cost function:

$$\gamma(\pi) = \theta_1 (\pi - 1) + \frac{\theta_2}{2} (\pi - 1)^2, \quad \pi > 1, \quad (19)$$

where  $\theta_1 \geq 0$  gives the minimal marginal funding cost for leverage in addition to the risk-free rate and  $\theta_2 \geq 0$  captures the convexity of the funding cost (i.e. the marginal increase of the marginal funding cost). The *marginal* funding cost function is linearly increasing in asset-capital ratio, in that  $\gamma'(\pi) = \theta_1 + \theta_2(\pi - 1)$  for  $\pi > 1$ . The optimal leverage ratio is thus given by

$$\pi^* = 1 + \frac{\alpha - \theta_1}{\theta_2}, \quad \text{if } \alpha > \theta_1. \quad (20)$$

It is immediate to see  $\pi^* > 1$  under the condition that alpha is sufficiently large to justify leverage, i.e.  $\alpha > \theta_1$ . For the rest of the analysis, we will focus on this interesting case where optimal leverage is  $\pi^* > 1$ , i.e.  $\alpha > \theta_1$ .

The expected excess return under optimal leverage (netting out the financing costs), i.e. optimally levered alpha is given by

$$\alpha^* = \alpha \left[ 1 + \frac{(\alpha - \theta_1)^2}{2\alpha\theta_2} \right]. \quad (21)$$

With optimal leverage, the net increase of alpha from  $\alpha$  to  $\alpha^*$  is thus given by

$$\Delta\alpha \equiv \alpha^* - \alpha = \frac{(\alpha - \theta_1)^2}{2\theta_2}, \quad (22)$$

which accounts for the net benefits of leverage (in flow terms) with no incentive problems.

The valuation formula for  $V^*(W)$  can be interpreted as a version of the Gordon dividend-growth formula. In the absence of payouts, the expected growth rate for the AUM is  $(r + \alpha^*)$ , where  $\alpha^*$  is the manager's ability to generate expected excess returns (using the optimal investment strategy  $\pi^*$  for the fund and after we net out the financing costs). Subtracting the total payout rate from the fund to the manager and investors is  $(c + \delta + \lambda)$ , we obtain the *net* expected growth rate of the fund,  $(r + \alpha^*) - (c + \delta + \lambda)$ . The appropriate discount rate is the risk-free rate because the fund's strategy is risk adjusted already. The instantaneous payout is  $(c + \delta + \lambda)W$ . Using the Gordon dividend-growth formula, we thus have

$$V^*(W) = \frac{(c + \delta + \lambda)W}{r - (r + \alpha^* - (c + \delta + \lambda))} = \frac{c + \delta + \lambda}{c + \delta + \lambda - \alpha^*}W. \quad (23)$$

Using essentially the same arguments to value management fees and investors' payoff, we obtain formulae (13) and (16) for  $M^*(W)$  and  $E^*(W)$ , respectively.

Our model features managerial skills ( $\alpha > 0$  and hence  $\alpha^* > 0$ ), and therefore, the manager captures rents in equilibrium giving rise to  $V^*(W) > W$ . Moreover, in equilibrium, we expect that investors break even in competitive markets, i.e.  $E^*(W) = W$ . Using (16), we set the equilibrium management fee  $c = \alpha^*$ . Consequently, using (13), we obtain that the equilibrium value of management fees is  $F^*(W) = \alpha^*W/(\delta + \lambda)$ .

We use the lower case to denote the value function in the corresponding upper case, per unit of high-water mark  $H$ . For example,  $f^*(w) = F^*(W)/H = \alpha^*w/(\delta + \lambda)$  and  $e^*(w) = E^*(W)/H = w$ .

While our benchmark model is set up for the case with  $k = 0$ , it also applies to settings where the high-water mark  $H$  approaches infinity ( $H \rightarrow \infty$ ). For that case, the incentive fees are effectively out of the money, and hence the manager only collects the management fees, and the solution is effectively the same as the benchmark solution with  $k = 0$ .

We next turn to the general setting where the manager collects both management and incentive fees. The high-water mark plays a fundamental role in leverage choices and valuation, resulting in imperfect alignment of incentives between the manager and investors.

## 4 Model solution: The general case with HWM

We solve the manager’s optimization problem using dynamic programming. First, we study the manager’s behavior and implied valuation when the AUM is below the high-water mark (i.e. in the interior region  $W < H$ ). Second, we analyze the manager’s behavior when the evolution of AUM leads to the new high-water mark. Third, we use the homogeneity property of our model to solve for the manager’s optimal investment strategy and various value functions for managers and investors.

**The manager’s intertemporal decision problem when  $W < H$ .** When the fund’s AUM is below its high-water mark ( $W < H$ ), instantaneously the manager will not receive the incentive fees. The fund is in the “interior” region where the option component (incentive fees) is in the very short term inactive, and the manager only collects management fees. However, being forward looking, he chooses dynamic investment strategy  $\pi$  to maximize  $F(W, H)$ , the present value of his total fees. Using the principle of optimality, we have the following Hamilton-Jacobi-Bellman (HJB) equation in the interior region:

$$(r + \lambda)F(W, H) = \max_{\pi} cW + [\pi(\mu - r) + (r - \gamma(\pi) - \delta - c)]WF_W(W, H) \quad (24)$$

$$+ \frac{1}{2}\pi^2\sigma^2W^2F_{WW}(W, H) + (g - \delta)HF_H(W, H).$$

The left side of (24) elevates the discount rate from the interest rate  $r$  to  $r + \lambda$  to reflect the stochastic liquidation of the fund. The first term on the right side of (24) gives the management fees in flow terms,  $cW$ . The second and third terms give the drift (expected change) effect and the volatility effect of AUM  $W$  on the value of total fees  $F(W, H)$ , respectively. Finally, the last term on the right side of (24) describes the effect of the high-water mark

$H$  change on  $F(W, H)$  in the interior region ( $W < H$ ). The manager optimally chooses the dynamic investment strategy  $\pi$  to equate the two sides of (24). Next, we analyze the properties of  $F(W, H)$  when AUM equals high-water mark and moves along the boundary  $W = H$ .

**The manager's intertemporal decision problem when  $W = H$ .** Our reasoning for the boundary behavior ( $W = H$ ) essentially follows GIR and Panageas and Westerfield (2009). Start with  $W = H$ . A positive return shock increases the AUM from  $W = H$  to  $H + \Delta H$ . The present value of total fees for the manager is then given by  $F(H + \Delta H, H)$  before the high-water mark adjusts. Immediately after the positive investment shock, the high-water mark adjusts to  $H + \Delta H$ . The manager then collects the incentive fees in flow terms  $k\Delta H$ , and consequently the AUM is lowered from  $H + \Delta H$  to  $H + \Delta H - k\Delta H$ . The manager's present value of total fees is then equal to  $F(H + \Delta H - k\Delta H, H + \Delta H)$ . Using the continuity of  $F(\cdot, \cdot)$  before and after the adjustment of the high-water mark, we have

$$F(H + \Delta H, H) = k\Delta H + F(H + \Delta H - k\Delta H, H + \Delta H). \quad (25)$$

By taking the limit  $\Delta H$  to zero and using the Taylor's rule, we obtain the following result:

$$kF_W(H, H) = k + F_H(H, H). \quad (26)$$

The above condition can be viewed as a value-matching condition for the manager when moving on the boundary  $W = H$ . By using essentially the same logic, we obtain the following boundary conditions for the PV of management fees  $M(W, H)$  and the PV of incentive fees  $N(W, H)$  at the boundary  $W = H$ :  $kM_W(H, H) = M_H(H, H)$  and  $kN_W(H, H) = k + N_H(H, H)$ .

Finally, we turn to the left boundary condition. When the fund runs out of assets ( $W = 0$ ), there is no more AUM in the future and hence no fees:  $F(0, H) = 0$ . In Section 6, we extend our model to allow investors to have more control rights. For example, investors may prevent the AUM from falling below a certain threshold by liquidating the fund. We show that this lower liquidation boundary has significant effects on the manager's dynamic investment strategies and valuation.

The tractability of our model solution rests on the homogeneity property. That is, if we double AUM  $W$  and the high-water mark  $H$  at the same time, the present value of total

fees  $F(W, H)$  will also double. The effective state variable is the ratio between AUM  $W$  and high-water mark  $H$ :  $w = W/H$ . Whenever applicable, we will use the lower case to denote the corresponding variable in the upper case scaled by contemporaneous high-water mark  $H$ . For example,  $f(w) = F(W, H)/H$ . The following theorem summarizes the main results on  $\pi(w)$  and value  $F(W, H)$ .

**Theorem 1** *The present value of total fees for the manager,  $F(W, H)$ , is homogenous of degree one in  $W$  and  $H$ , i.e.  $F(W, H) = f(w)H$ , where  $f(w)$  solves the following ordinary differential equation (ODE):*

$$(r + \lambda - g + \delta)f(w) = cw + [\alpha\pi(w) + r - \gamma(\pi(w)) - g - c]wf'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2f''(w), \quad (27)$$

where the optimal investment strategy  $\pi(w)$  solves the following equation:

$$\pi(w) = \frac{\alpha - \gamma'(\pi(w))}{\sigma^2\psi(w)}, \quad (28)$$

and  $\psi(w)$  is given by

$$\psi(w) = -\frac{wf''(w)}{f'(w)}. \quad (29)$$

The ODE (27) for  $f(w)$  is solved subject to the following boundary conditions:

$$f(0) = 0, \quad (30)$$

$$f(1) = (k + 1)f'(1) - k. \quad (31)$$

The formula (29) for  $\psi(w)$  is analogous to the definition of coefficient of relative risk aversion in consumer theory if we interpret  $f(w)$  as the counterpart to the consumer's value function. We may intuitively refer  $\psi(w)$  as the manager's effective risk attitude. However,  $f(w)$  may be convex due to optionality induced by the high-water mark. We will discuss the properties of  $\psi(w)$  in detail in Section 5.

With quadratic financing cost function (19), we have the following investment strategy:

$$\pi(w) = \left(1 + \frac{\alpha - \theta_1}{\theta_2}\right) \frac{1}{1 + \theta_2^{-1}\sigma^2\psi(w)} = \pi^* \frac{1}{1 + \theta_2^{-1}\sigma^2\psi(w)}, \quad (32)$$

where  $\pi^* = 1 + (\alpha - \theta_1)/\theta_2$  is the time-invariant optimal investment strategy without incentive fees ( $k = 0$ ). Incentive fees induce convexity (and also potentially concavity) of  $f(w)$  and hence accordingly influences the manager's investment strategy.

We now turn to the implied dynamic valuation for various components of the fund. We summarize the main results in the following proposition.

**Proposition 2** *Given investment strategy  $\pi(w)$ , the dynamics of  $w = W/H$  is given by:*

$$dw(t) = \mu_w(w(t))dt + \sigma\pi(w(t))w(t)dB(t) - dJ(t), \quad (33)$$

where  $J$  is a pure jump process which sets  $w = 0$  upon the stochastic arrival of the jump with intensity  $\lambda$ , and  $\mu_w(w)$  is given by

$$\mu_w(w) = [\alpha\pi(w) + (r - \gamma(\pi(w)) - g - c)]w. \quad (34)$$

The value functions  $M(W, H)$ ,  $N(W, H)$ , and  $E(W, H)$  are all homogeneous with degree one in AUM  $W$  and the high-water mark  $H$ , i.e.  $M(W, H) = m(w)H$ ,  $N(W, H) = n(w)H$ ,  $E(W, H) = e(w)H$ , where  $m(w)$ ,  $n(w)$ ,  $e(w)$  solve the following ODEs respectively:

$$(r + \lambda - g + \delta)m(w) = cw + \mu_w(w)m'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2m''(w), \quad (35)$$

$$(r + \lambda - g + \delta)n(w) = \mu_w(w)n'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2n''(w), \quad (36)$$

$$(r + \lambda - g + \delta)e(w) = (\delta + \lambda)w + \mu_w(w)e'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2e''(w), \quad (37)$$

with the following boundary conditions:

$$m(0) = n(0) = e(0) = 0, \quad (38)$$

$$m(1) = (k + 1)m'(1), \quad (39)$$

$$n(1) = (k + 1)n'(1) - k, \quad (40)$$

$$e(1) = (k + 1)e'(1). \quad (41)$$

The total fund's value is  $V(W, H) = v(w)H$ , where  $v(w) = m(w) + n(w) + e(w) = f(w) + e(w)$ .

**Investors' voluntary participation.** In order for investors to voluntarily participate, we need to ensure that investors do not lose money in present value by investing with the manager, i.e.  $E(W_0, W_0) \geq W_0$  as given in (12). It is equivalent to require  $e(1) \geq 1$  due to the homogeneity property of  $E(W, H)$ . With perfectly competitive capital markets, the manager collects all the rents from their managerial skill, and therefore, we have the equilibrium outcome for the investors' value:

$$e(1) = 1. \quad (42)$$

We next analyze the implications of the model solution for this general case with HWM.



## 5 Results: compensation, leverage, and valuation

We now explore the effect of managerial compensation on leverage and valuation by using the results from Section 4.

**Parameter choices.** All rates are annualized and continuously compounded. For the baseline calculation, we choose the following parameter values: the interest rate  $r = 5\%$ , the payout rate to investors  $\delta = 5\%$ , the annual liquidation probability  $\lambda = 5\%$ , and target growth rate of the high-water mark  $H$  in the absence of withdrawal  $g = 5\%$  so that  $g - \delta = 0$ . As a result, the high-water mark  $H$  does not change unless AUM  $W$  reaches a new record, i.e.  $H$  is the running maximum of AUM  $W$ :  $H(t) = \max\{W(v) : v \in [0, t]\}$ , which only moves upward when the AUM  $W$  reaches a new all-time high level.

We choose the financing cost using the quadratic function given in (19). The implied marginal financing cost function is linearly increasing in  $\pi$ :  $\gamma'(\pi) = \theta_1 + \theta_2(\pi - 1)$  for  $\pi \geq 1$  and  $\gamma'(\pi) = 0$  for  $0 \leq \pi \leq 1$ . The parameter  $\theta_1$  is the intercept in the linear marginal financing cost function  $\gamma'(\pi)$  and gives the minimal level of the marginal financing cost for any borrowed amount. The curvature parameter  $\theta_2$  gives the slope of the marginal cost of financing as a function of the total borrowed amount per AUM:  $(\pi - 1)$ . In our example, we set  $\theta_1 = 0.25\%$  and  $\theta_2 = 0.5\%$ . The fund thus pays extra 25 basis points in addition to the risk-free rate 5% on each borrowed dollar. The interest rate for the first dollar that the fund borrows is thus equal to 5.25%. For additional borrowing, the fund pays higher rate. For example, with  $\pi = 5.5$ , the fund pays the marginal funding cost  $\theta_2 \times (\pi - 1) = 0.5\% \times (5.5 - 1) = 2.25\%$  on the last dollar borrowed. On top of the rate 5.25%. This puts the marginal financing cost for the last borrowed dollar with  $\pi = 5.5$  at 7.5%.

We consider the manager with an un-levered expected excess return  $\alpha = 2.5\%$  and an annual volatility of unlevered risky investment strategy to be  $\sigma = 10.7\%$ . Before we delve into the analysis of the case with incentive fees, we first summarize the benchmark results from Section 3 under the contract without incentive fees ( $k = 0$ ).

### 5.1 High-water mark, leverage $\pi(w)$ , and managerial rents $f(w)$

**The benchmark with no incentive fees ( $k = 0$ ).** The equilibrium management fee is shown to be  $c = 7.56\%$ . Investors break even under perfectly competitive capital markets:

$e(1) = 1$ . The optimal leverage is time-invariant and is equal to  $\pi^* = 5.5$ . For each dollar of AUM, the fund borrows 4.5 dollars and invests 5.5 dollars in the risky positive-alpha asset. As a result, the fund enhances its alpha from unlevered  $\alpha = 2.5\%$  to net-of-fees levered  $\alpha^* = 7.56\%$  via optimal leverage. The manager collects all the surplus by receiving  $c = \alpha^* = 7.56\%$  of AUM via management fees provided that the fund is in operation. The dashed line in Figure 1 plots the time-invariant optimal leverage  $\pi^* = 1 + (\alpha - \theta_1)/\theta_2 = 5.5$ .

**The effect of incentive fees ( $k > 0$ ).** The solid line in Figure 1 plots  $\pi(w)$  as a function of  $w$  for the case with  $k = 20\%$  and  $c = 2\%$  (a typical 2-20 compensation contract). Investors break even at the moment of participation, i.e.  $e(1) = 1$ . Compared with the benchmark ( $k = 0$ ), the manager invests more in its positive-alpha technology when  $k > 0$ :  $\pi(w) > \pi^*$  for  $w > 0$ . Optionality embedded in incentive fees ( $k > 0$ ) encourages *excessive* risk taking.

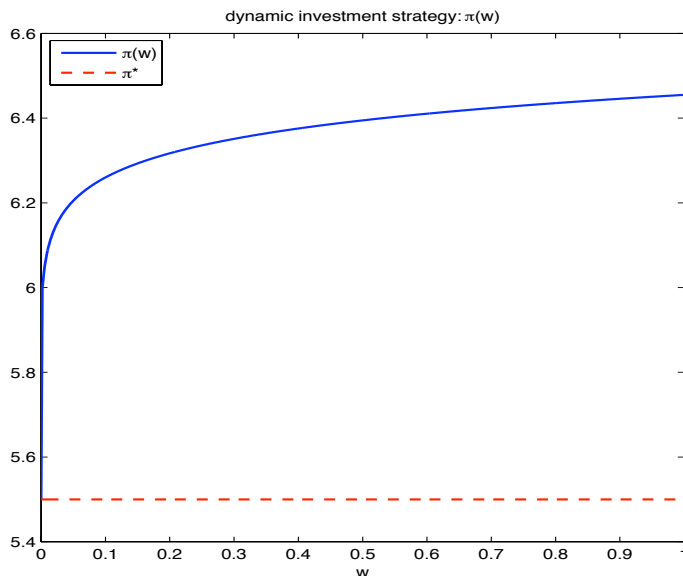


Figure 1: **Dynamic investment strategy  $\pi(w)$ .** The solid line corresponds to the 2-20 managerial compensation contract:  $c = 0.02$ ,  $k = 0.2$ ; and the dashed line corresponds to the benchmark contract with no incentive fees:  $k = 0$ ,  $c = 7.56\%$ . Both contracts satisfy  $e(1) = 1$ . The other parameter values are  $r = 0.05$ ,  $\lambda = 0.05$ ,  $g = 0.05$ ,  $\delta = 0.05$ ,  $\theta_1 = 0.25\%$ , and  $\theta_2 = 0.5\%$ .

In our example, under a 2-20 compensation contract (i.e. 2% management and 20% incentive fees),  $\pi(1) = 6.45$ , about 17% higher than the no-incentive-fee benchmark level of optimal leverage  $\pi^* = 5.5$ . The excessive risk-taking incentive under 2-20 vanishes at  $w = 0$

because the optionality due to incentive fees disappears at  $w = 0$ . For  $0 \leq w \leq 1$ , the manager's risk-seeking incentive increases with  $w$ , i.e.  $\pi'(w) > 0$ .

The left panel of Figure 2 plots managerial rents  $f(w)$  under the 2-20 contract and compares with  $f^*(w)$  for the  $k = 0$  benchmark. Greater risk taking induced by incentive fees lowers total managerial rents:  $f(w) < f^*(w)$ . However, quantitatively, this effect is small in our example:  $f(1) = 0.737$ , which is slightly smaller than  $f^*(1) = 0.756$ .

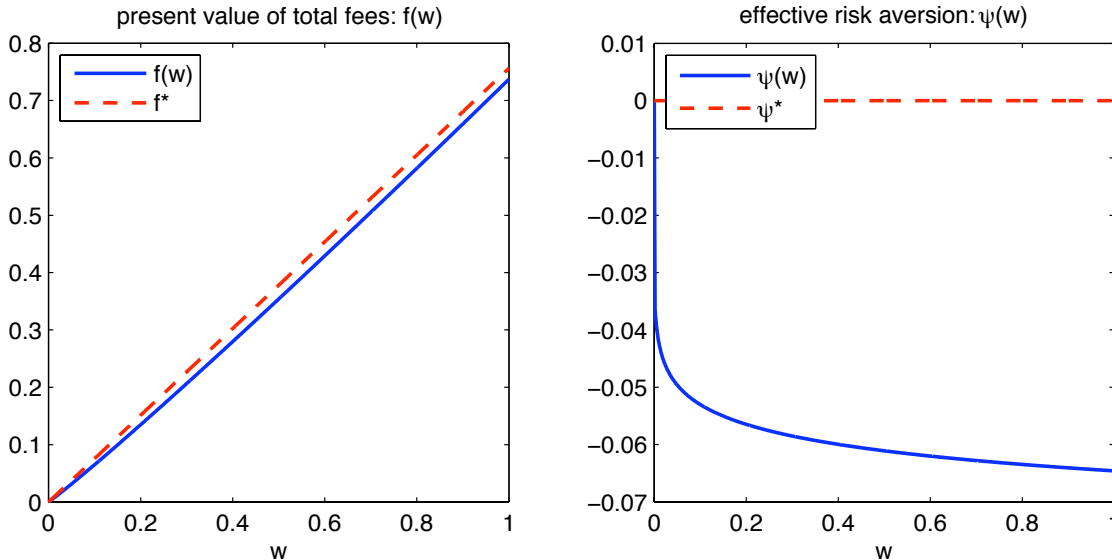


Figure 2: **The manager's present value of total fees  $f(w)$  and his "effective" risk attitude:  $\psi(w) = -wf''(w)/f'(w)$ .** The manager's value  $f(w)$  is convex, indicating that he engages in excessive risk seeking ( $\psi(w) < 0$ ) by taking on more leverage with incentive fees ( $k > 0$ ) than without ( $k = 0$ ). However, the magnitude is small.

The right panel of Figure 2 plots the manager's risk attitude  $\psi(w) = -wf''(w)/f'(w)$ , which is defined in (29). In the benchmark with  $k = 0$ , we have  $\psi^*(w) = 0$  for all  $w$ , i.e. the manager is risk neutral. When  $k > 0$ , the risk-neutral manager takes on excessive risk. Using the optimal leverage formula (32), we see that  $\pi(w) > \pi^*$  if and only if  $\psi(w) < 0$  (i.e.  $f(w)$  is convex). Moreover, leverage  $\pi(w)$  increases with  $w$  because the manager effectively becomes more risk loving ( $\psi(w) < 0$  and  $|\psi(w)|$  increases with  $w$ ). However, quantitatively, excessive risk taking in our baseline case has insignificant effects on  $f(w)$  and  $\psi(w)$ .

**Marginal value of AUM  $F_W(W, H)$  and marginal value of high-water mark  $F_H(W, H)$ .**

Using the homogeneity property of  $F(W, H)$ , we have the following analytical results for the marginal value of AUM on the PV of total fees  $F(W, H)$ ,  $F_W(W, H)$ , and the marginal value of the HWM on the PV of total fees  $F(W, H)$ ,  $F_H(W, H)$ :

$$F_W(W, H) = f'(w), \quad (43)$$

$$F_H(W, H) = f(w) - wf'(w). \quad (44)$$

For the benchmark with  $k = 0$ , the manager collects a rent of 75.6 cents in PV for each dollar of AUM. The dashed horizontal line in the left panel of Figure 3 depicts the constant marginal value of AUM:  $F_W^*(W) = f'_w(w) = 0.756$  for all  $w$ . The solid line in the left panel

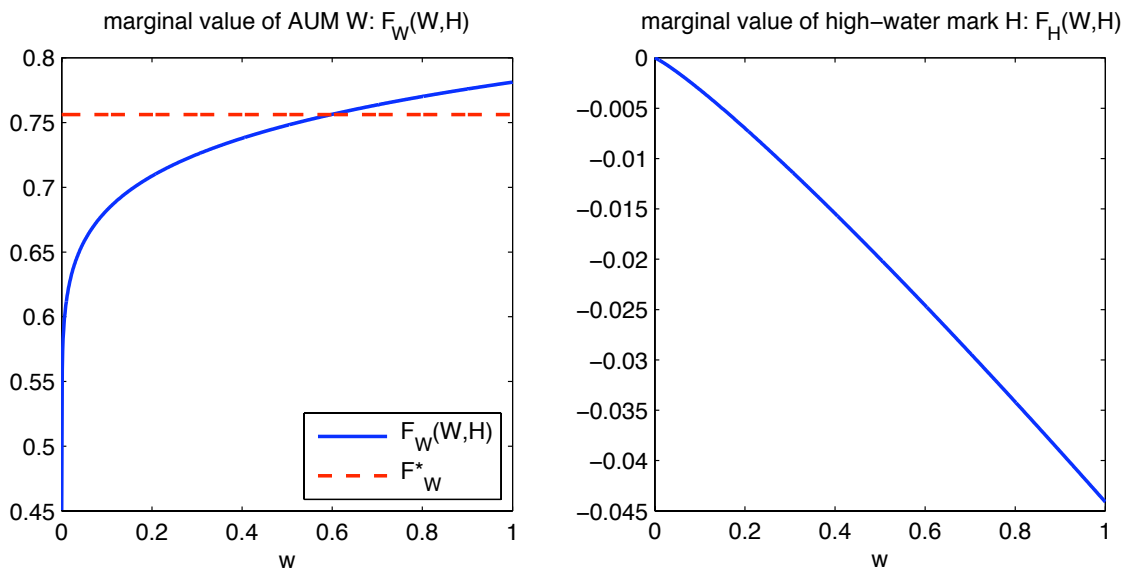


Figure 3: **The marginal value of AUM  $W$ :  $F_W(W, H)$  and the marginal value of high-water mark  $H$ :  $F_H(W, H)$ .**

of Figure 3 plots  $F_W(W, H) = f'(w)$  for the case with incentive fees ( $k > 0$ ). First,  $f'(w)$  is positive and increasing in  $w$ . Second,  $F(W, H)$  is convex in AUM  $W$  (i.e.  $f''(w) > 0$ ) due to the same call option feature embedded in incentive fees. Third, the marginal value of AUM  $F_W$  can be either higher or lower than the no-incentive-fee ( $k = 0$ ) benchmark  $F_W^*(W) = 0.756$ . If  $w$  is sufficiently high, the marginal value of AUM  $F_W$  is higher than 0.756, the level for the  $k = 0$  benchmark because of the high powered incentives from the

fee structure. For example,  $F_W$  reaches \$0.78 at  $w = 1$ :  $f'(1) = 0.78$ . For sufficiently low  $w$ , the marginal value of AUM  $F_W$  is lower than the benchmark level:  $F^*(W) = 0.756$ . For example, when  $w = 0$  (say due to  $H \rightarrow \infty$ ), the incentive fee is completely out of the money and  $F_W(0, H) = f'(0) = c/(\delta + \lambda + c - \alpha^*) = 0.02/(0.05 + 0.05 + 0.02 - 0.0756) = 0.45$ .

The right panel of Figure 3 plots  $F_H(W, H)$ , the marginal effect of the high-water mark on the PV of total fees  $F(W, H)$ . The high-water mark  $H$  is effectively the strike price of incentive fees, which is stochastic and endogenous. Increasing  $H$  lowers the present value of total fees  $F(W, H)$ , i.e.  $F_H(W, H) < 0$ .

## 5.2 Valuing various components for the manager and investors

**Decomposing managerial rents  $F(W, H)$  into the value of management fees  $M(W, H)$  and the value of incentive fees  $N(W, H)$ .** The upper and lower left panels of Figure 4 plot the scaled PVs of management fees  $m(w)$  and of incentive fees  $n(w)$ , respectively. Not surprisingly, both  $m(w)$  and  $n(w)$  increase with  $w$ . More importantly, the incentive fee accounts for the major portion of managerial rents:  $n(1) = 0.54$  which is 73 percentage of  $f(1) = 0.74$  while  $m(1) = 0.20$  gives the remaining managerial rents.

The upper and lower center panels of Figure 4 plot  $M_W(W, H) = m'(w)$  and  $N_W(W, H) = n'(w)$ , respectively. First,  $n'(w)$  is increasing in  $w$ . That is, the value of incentive fees  $N(W, H)$  is convex in AUM  $W$  due to the embedded optionality of incentive fees. Second,  $m'(w)$  is decreasing in  $w$ . That is, the value of management fees  $M(W, H)$  is concave in AUM  $W$ . Incentive fees crowd out management fees, and the payment of incentive fees lowers the AUM. The substitution of incentive fees for management fees effectively makes the value of management fees  $M(W, H)$  concave in  $W$ .<sup>8</sup>

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<sup>8</sup>Of course, there is an overall risk-taking effect induced by the incentive fees, i.e. the value of total fees  $F(W, H)$  is convex in  $W$ . The crowding-out effect is stronger than the overall risk-taking effect which makes  $F(W, H)$  and  $N(W, H)$  convex in  $W$  but  $M(W, H)$  concave in  $W$ .

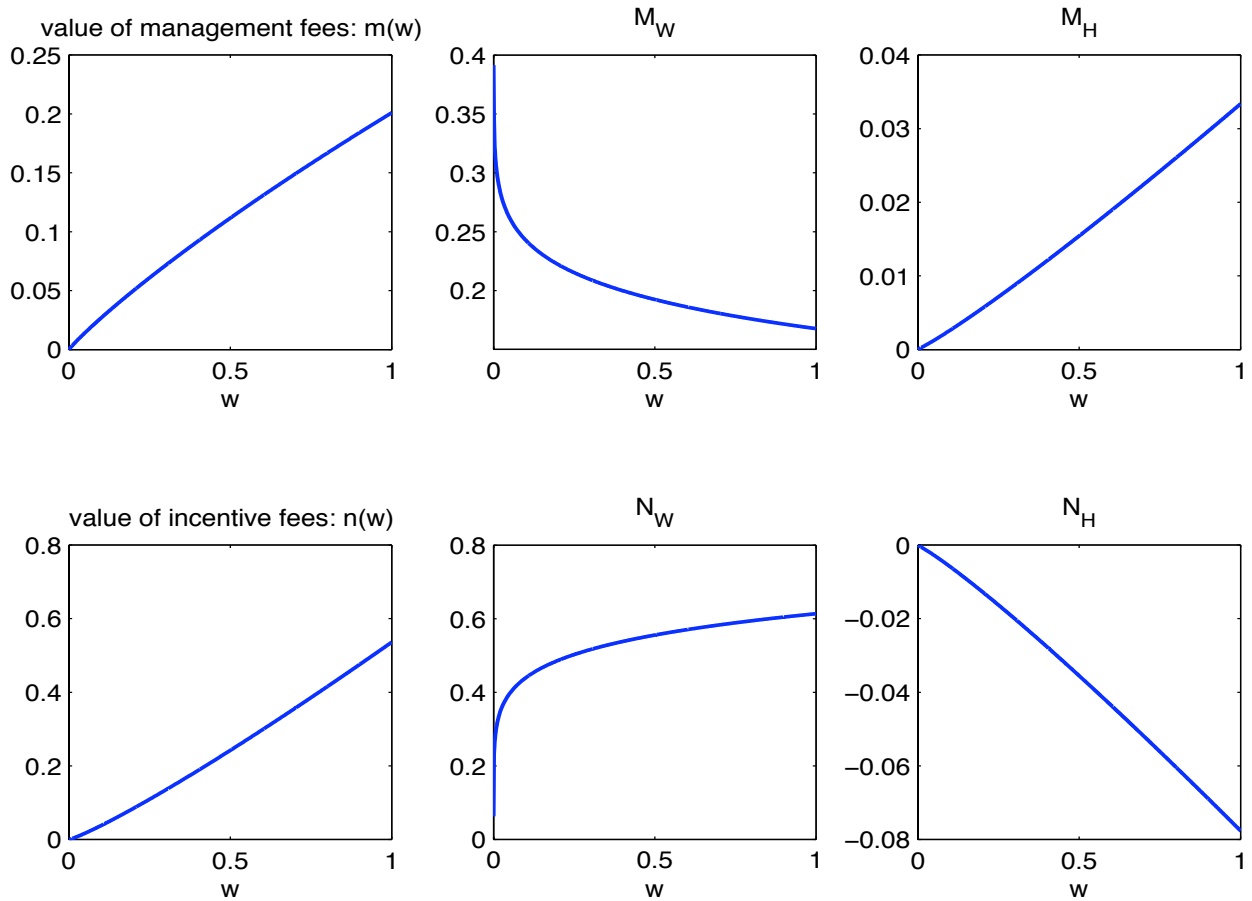


Figure 4: **Decomposing managerial rents  $f(w)$  into the present value of management fees  $m(w)$  and the present value of incentive fees  $n(w)$ :  $f(w) = m(w) + n(w)$ .**

We now turn to the effect of  $H$  on  $M(W, H)$  and  $N(W, H)$ . Intuitively, the higher the high-water mark  $H$ , the less likely the manager collects the incentive fees and hence  $N_H(W, H) < 0$ . The higher the high-water mark, the less likely the manager draws on the incentive fees and hence the higher the value of management fees  $M(W, H)$ , i.e.  $M_H(W, H) > 0$ . The upper and lower right panels in Figure 4 plot  $M_H(W, H)$  and  $N_H(W, H)$ , respectively.

**Investors' value of payoff  $E(W, H)$  and total value of the fund  $V(W, H)$ .** The upper row of Figure 5 plots scaled investors' value  $e(w)$ , and the marginal effects of AUM and high-water mark on investors' value:  $E_W(W, H)$  and  $E_H(W, H)$ . Investors' value  $e(w)$  is increasing and concave because investors are short in the option embedded in incentive fees, whose value is convex in AUM. The higher the high-water mark, the less valuable the

incentive fees and the higher investors' value, i.e.  $E_H(W, H) > 0$ .

The lower row of Figure 5 plots the scaled total fund value  $v(w) = V(W, H)/H$ ,  $V_W(W, H)$ , and  $V_H(W, H)$ . While investors' value  $e(w)$  is concave and the manager's value  $f(w)$  is convex, the sum of the two,  $v(w) = f(w) + e(w)$ , is concave. Paying the manager incentive fees lowers AUM and the fund in net is short of an incentive option. Finally, the total fund value  $V(W, H)$  increases with high-water mark because the gain to investors from a higher  $H$  outweighs the corresponding loss to the manager.

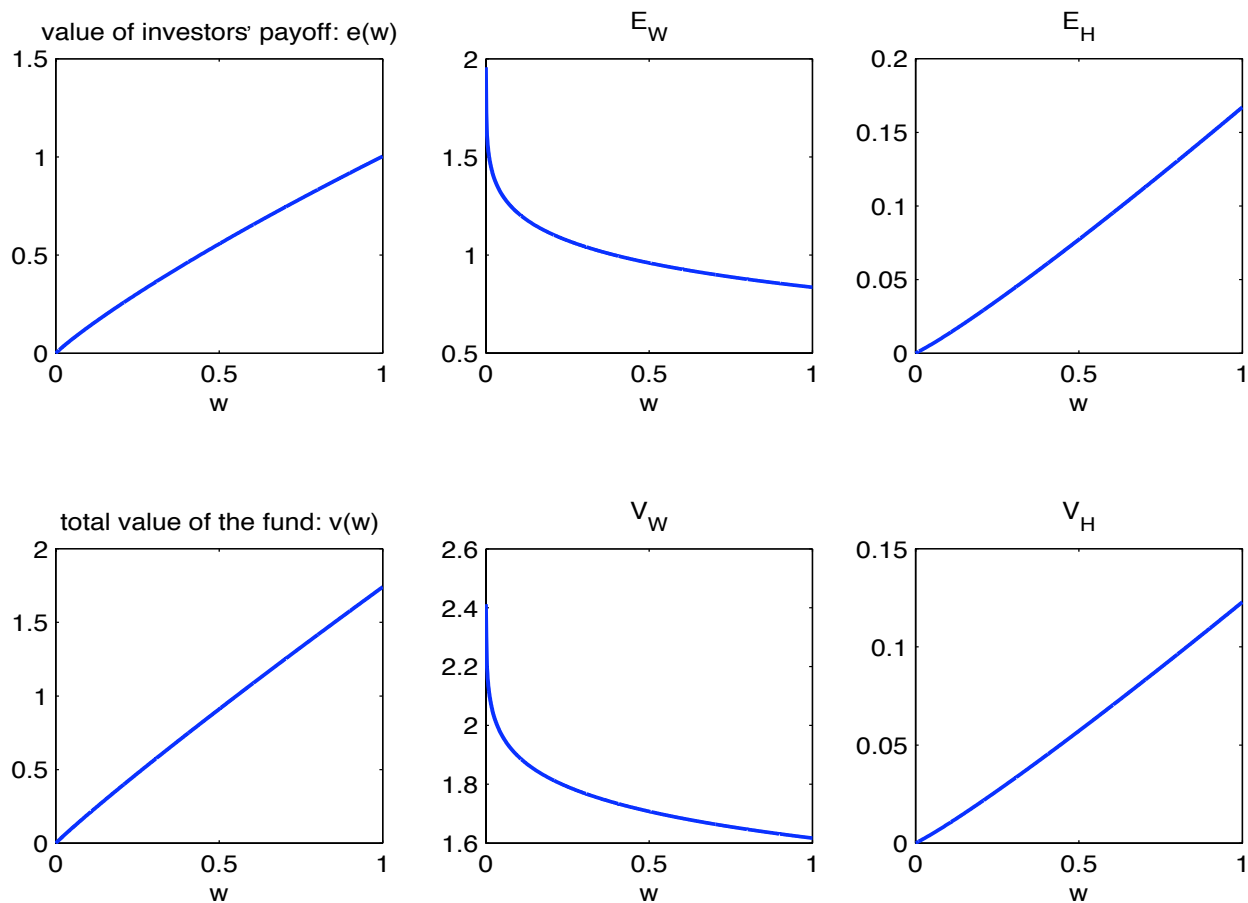


Figure 5: Scaled investors' value  $e(w)$  and the total fund value  $v(w)$ .

## 6 Fund liquidation and investment strategy

For simplicity, we have so far intentionally kept our analysis parsimonious. One crucial simplification is that the fund has no performance-triggered withdrawal. However, in reality,

when the fund performance deteriorates sufficiently, investors may withdraw and sometimes liquidate the fund in order to protect themselves against downside risk. Investors' liquidation rights are sometimes contracted.

We will show that investors' option to liquidate the fund significantly changes the economics of managerial leverage and also has important quantitative implications. We model investors' liquidation rights by allowing the investor to liquidate the fund if the manager performs sufficiently poorly. More specifically, assume that the fund is liquidated whenever AUM  $W$  falls below a fixed fraction  $b$  of the high-water mark  $H$ . GIR make the same assumption on the liquidation boundary but the AUM evolution is exogenously given.

The manager is now potentially averse to performance-linked withdrawal/liquidation because the manager loses both management and incentive fees if the fund is liquidated.<sup>9</sup> The manager rationally manages the fund's leverage to influence the AUM evolution and hence the liquidation likelihood. As we will show, he may delever the fund when it is sufficiently close to the liquidation boundary in order to preserve his future fees. We summarize the main results in Theorem 2 and Proposition 2 as follows:

**Theorem 2** *The manager's present value of total fees,  $F(W, H)$ , is homogenous of degree one in  $W$  and  $H$  and is given by  $F(W, H) = f(w)H$ , where  $f(w)$  solves the following ODE:*

$$(r + \lambda - g + \delta)f(w) = (c + \delta + \lambda)w + \mu_w(w)f'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2f''(w), \quad (45)$$

where  $\mu_w(w)$  is given by (34). The ODE (45) is solved subject to the boundary conditions:

$$f(b) = 0, \quad (46)$$

$$f(1) = (k + 1)f'(1) - k. \quad (47)$$

Moreover,  $f(w)$  is continuously differentiable across three regions, which are optimally determined via endogenous cutoff points  $w_L$  and  $w_R$ , where  $b \leq w_L \leq w_R \leq 1$ . That is,

$$f(w_L-) = f(w_L+), f(w_R-) = f(w_R+), f'(w_L-) = f'(w_L+), f'(w_R-) = f'(w_R+). \quad (48)$$

The corresponding optimal investment strategy  $\pi(w)$  in these three regions are as follows:

1. If  $w$  is sufficiently high ( $w > w_R$ ),  $\pi(w)$  solves (28) with the property  $\pi(w) > 1$ .

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<sup>9</sup>See Dai and Sundaresan (2010) on this point in a model without high-water mark.



2. If  $w$  is in the intermediate region (i.e.  $w_L \leq w \leq w_R$ ),  $\pi(w) = 1$ .

3. If  $w$  is sufficiently low ( $b \leq w < w_L$ ), the optimal investment strategy  $\pi(w)$  is given by

$$\pi(w) = \frac{\alpha}{\sigma^2 \psi(w)} \leq 1. \quad (49)$$

**The manager’s optimal investment strategy  $\pi(w)$  with liquidation boundary  $b > 0$ .** Figure 6 plots the manager’s optimal leverage  $\pi(w)$  with liquidation boundary  $b = 0.5$  and a “new” compensation contract (with  $c = 2.45\%$  and  $k = 20\%$ ) which ensures that investors break even ( $e(1) = 1$ ) with  $b = 0.5$ . Because higher liquidation boundary better protects investors, investors thus offer a more attractive contract ( $c = 2.45\%$  than  $c = 2\%$  holding  $k = 20\%$  fixed) to the manager with the same skill (i.e.  $\alpha = 2.5\%$ ). All other parameter values remain unchanged. Leverage  $\pi(w)$  is significantly lower for all  $w$  with liquidation boundary (i.e.  $b = 0.5$  vs  $b = 0$ ). Dai and Sundaresan (2010) also show that hedge funds may use conservative levels of leverage if they properly recognize the short option positions due to contractual arrangements with investors and prime brokers.

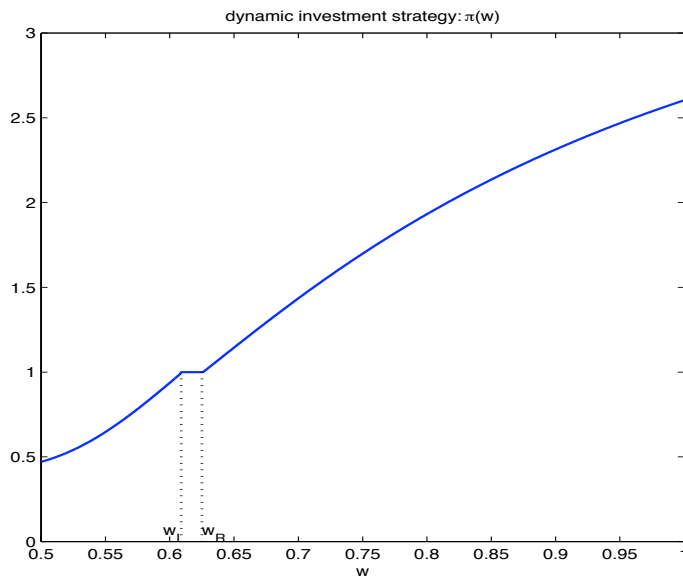


Figure 6: **Dynamic investment strategy  $\pi(w)$ : The case with liquidation boundary  $b = 0.5$ .** The managerial compensation contract  $(c, k)$  is given by  $(2.45\%, 20\%)$ . Investors break even:  $e(1) = 1$ . Other parameter values are:  $r = 0.05$ ,  $\alpha = 0.025$ ,  $\sigma = 10.7\%$ ,  $\lambda = 0.05$ ,  $g = 0.05$ ,  $\delta = 0.05$ ,  $\theta_1 = 0.25\%$ , and  $\theta_2 = 0.5\%$ .

Unlike the case without liquidation ( $b = 0$ ), the optimal investment strategy  $\pi(w)$  differs across three regions depending on the value of  $w$ , and can sometimes be less than unity. Importantly, the cutoff points for the three regions of  $w$ ,  $w_L$  and  $w_R$ , are part of the solution for the manager’s optimization problem. For our example,  $w_L = 0.61$  and  $w_R = 0.63$ .

First, consider the case with sufficiently high  $w$  (i.e.  $w \geq 0.63$ ). The manager takes on leverage ( $\pi > 1$ ), but much lower than the level in the baseline case with  $b = 0$ . For example, even at  $w = 1$ , the leverage is only  $\pi(1) = 2.60$ , which is substantially lower than the level  $\pi^* = 6.45$  with  $b = 0$ . The embedded optionality due to incentive fees is substantially lower due to the presence of the liquidation boundary.

Second, consider the case where  $w$  is sufficiently close to the liquidation boundary  $b = 0.5$  (i.e.  $0.5 < w \leq 0.61$ ). In this case, survival is the manager’s primary concern. He rationally allocates some of its AUM to the risk-free asset ( $0 < \pi^*(w) < 1$ ) in order to lower the volatility of AUM return even though the risky investment strategy has positive alpha. Liquidation is costly because the manager loses future rents from the alpha generating technology. By reducing exposure to the risky positive alpha technology, the manager lower current payoffs in exchange for higher future rents. When  $w$  is sufficiently close to the liquidation boundary  $b = 0.5$ , the optimal investment satisfies  $\pi(w) < 1$ . The manager’s concern for survival leads him not only to delever but moreover to *save* some of the fund’s AUM in the risk-free asset.<sup>10</sup>

Third, for the intermediate range of  $w$  (i.e.  $0.61 < w < 0.63$ ), the manager holds all its AUM in the risky alpha generating technology ( $\pi = 1$ ) and no position in the risk-free asset. This reflects the compromise between capturing alpha and keeping the fund’s volatility from getting too high. Why is the optimal choice of  $\pi(w)$  unity for a range of  $w$  between 0.61 and 0.63? This is due to the assumption that the marginal financing cost is at least as large as  $\theta_1 = 0.25\% > 0$  even when the fund decides to borrow an infinitesimal amount:  $\lim_{\pi \rightarrow 1+} \gamma'(\pi) = \theta_1 = 0.25\%$ . For the range of  $0.61 \leq w \leq 0.63$ , the marginal benefit of leverage for the manager is positive, but is lower than 25 basis points in flow terms. As a result, choosing  $\pi = 1$  is optimal for  $0.61 < w < 0.63$ . We next turn to the manager’s value of total fees  $f(w)$  and his “effective” risk appetite  $\psi(w)$ .

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<sup>10</sup>By continuity, with sufficiently high  $b$ , the manager may choose finite leverage even in the absence of financing costs. Investors’ liquidation option acts as a disciplinary device which prevents the manager from taking on excessive leverage. This result is in stark contrast against the case with  $b = 0$  where leverage is unbounded in the absence of financing costs:  $\gamma(\pi) = 0$ .

**Managerial rents  $f(w)$  and risk attitude  $\psi(w)$ .** The left panel of Figure 7 plots  $f(w)$  when the liquidation boundary  $b = 0.5$ . Compared with the case without liquidation ( $b = 0$ ),  $f(w)$  is substantially lower. Liquidation shortens the duration of the fund's life and also reduces the manager's incentive to take on risk, both of which lower  $f(w)$ . With  $b = 0.5$ , the manager makes a rent of 29 cents for each dollar in AUM:  $f(1) = 0.29$ , which is substantially lower than a rent of 74 cents per dollar of AUM:  $f(1) = 0.74$  when  $b = 0$ . Note that this comparison has already accounted for the adjustment of managerial compensation from  $c = 2\%$  for  $b = 0$  to  $c = 2.45\%$  for  $b = 0.5$  to ensure competitive capital markets, i.e.  $e(1) = 1$ . Otherwise, the effect of liquidation is even greater if  $(c, k)$  is fixed at  $(2\%, 20\%)$ .

The right panel plots the manager's effective risk aversion  $\psi(w)$  when liquidation boundary is set at  $b = 0.5$ . Importantly, the sign of  $\psi(w)$  switches from minus to plus when we change the liquidation boundary from  $b = 0$  to  $b = 0.5$ ! That is, liquidation boundary turns the manager from a risk seeker (when  $b = 0$ ) to a risk averse one. Risk aversion is significant near liquidation:  $\psi(0.5) = 4.64$ . Even at  $w = 1$ , the manager continues to behave in an effectively risk-averse manner:  $\psi(1) = 0.49 > 0$ .

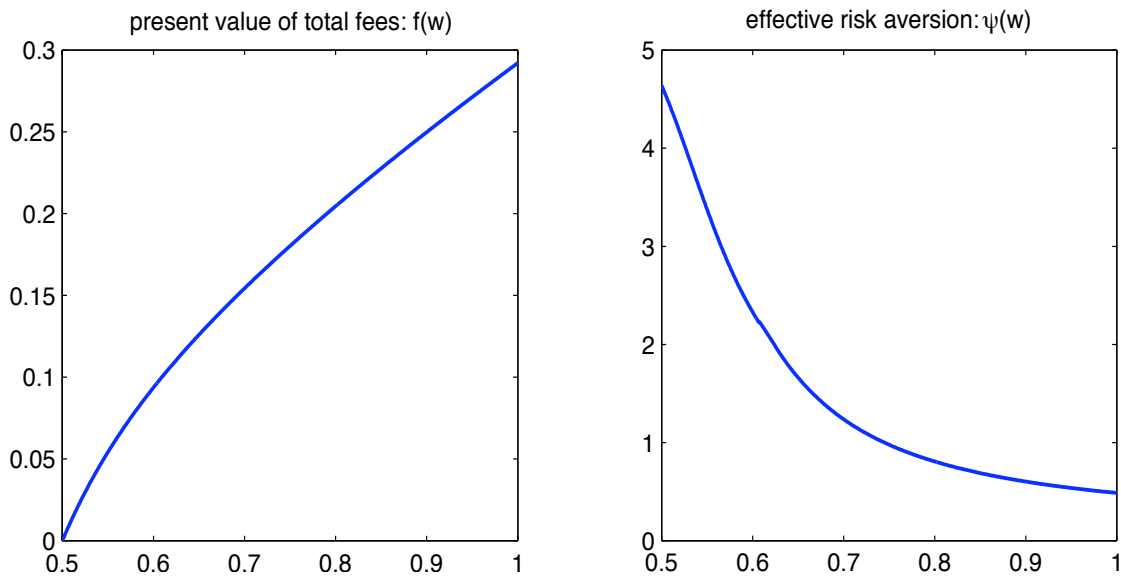


Figure 7: **The value of the manager's total fees  $f(w)$  and the manager's effective risk aversion:  $\psi(w) = -wf''(w)/f'(w)$ : The case with liquidation boundary  $b = 0.5$ .** The value of the manager's fees is substantially lower than that without liquidation boundary ( $b = 0$ ).

Our result runs against the conventional wisdom that incentive fees encourage excessive risk taking. The liquidation option fundamentally changes the managerial incentive to take on leverage. Indeed, the manager sometimes behaves too conservatively from the investors' perspective. This provides one explanation why, empirically, we see that investors sometimes require a lower bound for the fund's leverage. Managers may sometimes choose excessively low leverage primarily for survival and fee collection, which is detrimental to investors' welfare. Next, we characterize various value functions.

**Proposition 3** *The scaled value functions  $m(w)$ ,  $n(w)$ , and  $e(w)$  solve (35), (36) and (37), subject to the following boundary conditions:*

$$m(b) = n(b) = 0, \quad e(b) = b. \quad (50)$$

*These functions are continuously differentiable, and satisfy the following conditions at the endogenously determined cutoff points  $w_L$  and  $w_R$  given in Theorem 2:*

$$\begin{aligned} m(w_L-) &= m(w_L+), \quad m(w_R-) = m(w_R+), \quad m'(w_L-) = m'(w_L+), \quad m'(w_R-) = m'(w_R+), \\ n(w_L-) &= n(w_L+), \quad n(w_R-) = n(w_R+), \quad n'(w_L-) = n'(w_L+), \quad n'(w_R-) = n'(w_R+), \\ e(w_L-) &= e(w_L+), \quad e(w_R-) = e(w_R+), \quad e'(w_L-) = e'(w_L+), \quad e'(w_R-) = e'(w_R+). \end{aligned}$$

*The upper boundary conditions at  $w = 1$  remain the same as in Proposition 2, i.e. conditions (39), (40), and (41), respectively. The total fund's value for both investors and the manager is  $V(W, H) = v(w)H$ , where  $v(w) = m(w) + n(w) + e(w) = f(w) + e(w)$ .*

Figure 8 plots the PV of management fees  $m(w)$ , the PV of incentive fees  $n(w)$ , investors' value  $e(w)$ , and the total fund value  $v(w)$  with  $b = 0.5$ . Not surprisingly, we also find the PV of management fees  $m(w)$  is increasing and concave, the PV of incentive fees  $n(w)$  is increasing and convex, and the value of total fund  $v(w)$  is increasing and concave in  $w$ . Perhaps somewhat surprisingly, the value of investors' value  $e(w)$  is convex in  $w$  for  $0.5 \leq w \leq 0.61$  and concave in  $w$  for  $0.61 \leq w \leq 1$ . The concavity for the right region is intuitive, similar to what we have for the baseline with  $b = 0$ . The convexity on the left is due to the option value of liquidation, which we do not have in the baseline case with  $b = 0$ . Quantitatively, the fund value substantially drops with a liquidation boundary  $b = 0.5$  than without:  $v(1) = 1.29$  with  $b = 0.5$  versus  $v(1) = 1.74$  with  $b = 0$ .

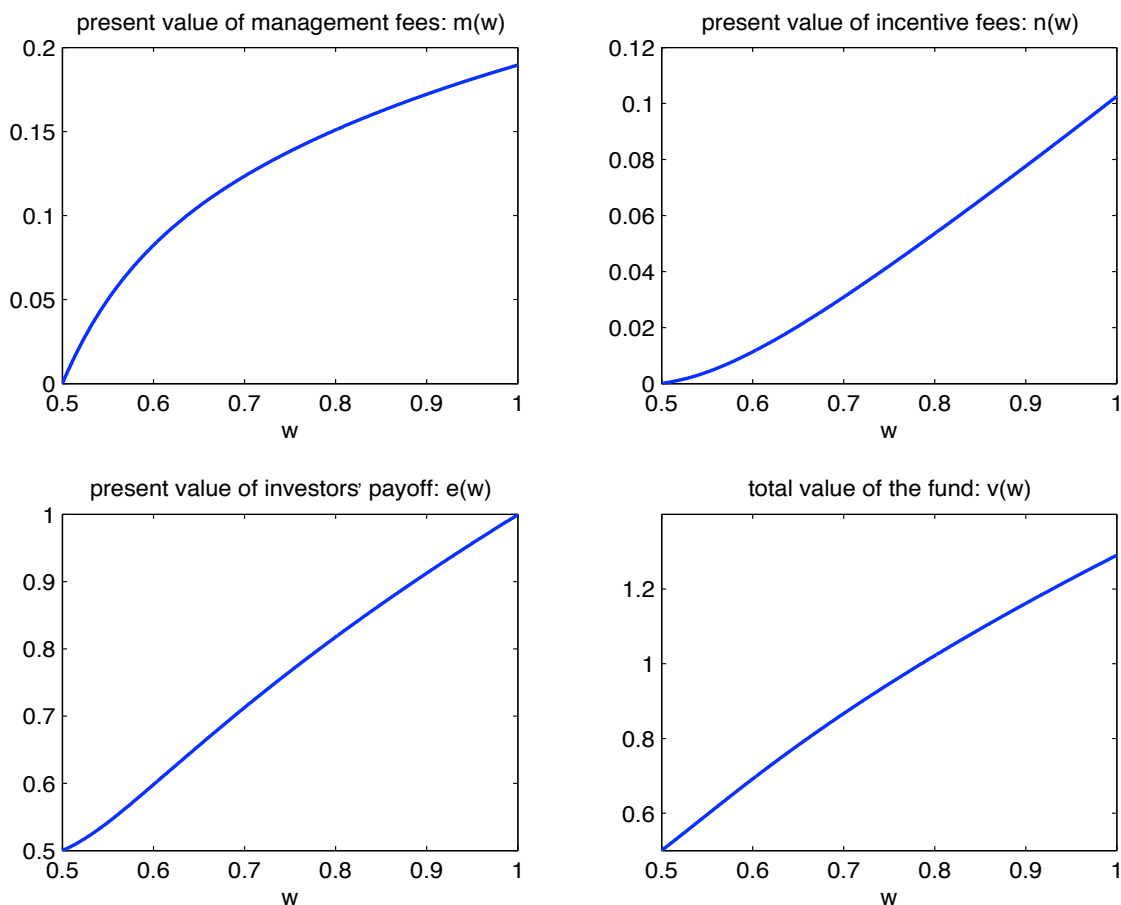


Figure 8: **The value of management fees  $m(w)$ , the value of incentive fees  $n(w)$ , investors' value  $e(w)$  and the total fund value  $v(w)$ : The case with lower liquidation boundary  $b = 0.5$ .**

To summarize, the liquidation boundary makes the manager's investment strategy much more conservative because preserving the fund as a going-concern is valuable for the manager. With a liquidation boundary, the value of fees  $f(w)$  is much lower and the manager behaves in a much more risk-averse way. Liquidation risk and the manager's response via leverage decisions are the first-order issues both conceptually and quantitatively.

## 7 Managerial ownership

Managers often have concentrated ownership in their funds for reasons such as aligning incentives and mitigating adverse selection. In equilibrium, rational investors take into

account managerial ownership in the fund when making investment decisions. We extend our model of Section 6 to incorporate managerial ownership and analyze the agency conflict.

Let  $\phi$  denote the managerial ownership in the fund. For simplicity, we assume that  $\phi$  is constant over time. Let  $Q(W, H)$  denote the manager's total value, which includes both the value of total fees  $F(W, H)$  and  $\phi E(W, H)$ , the manager's share as an investor:

$$Q(W, H) = F(W, H) + \phi E(W, H). \quad (51)$$

The manager dynamically chooses the investment policy  $\pi(w)$  to maximize (51). The following theorem gives the optimal  $\pi(w)$  and the scaled manager's total value  $q(w) = f(w) + \phi e(w)$ .

**Theorem 3** *The manager's scaled total value  $q(w)$  solves the following ODE:*

$$(r + \lambda - g + \delta)q(w) = [c + \phi(\delta + \lambda)]w + \mu_w(w)q'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2q''(w), \quad (52)$$

where  $\mu_w(w)$  is given by (34). The ODE (52) is solved subject to the boundary conditions:

$$q(b) = \phi b, \quad (53)$$

$$q(1) = (k + 1)q'(1) - k. \quad (54)$$

Moreover,  $q(w)$  is continuously differentiable across three regions, which are optimally determined via endogenous cutoff points  $w_L$  and  $w_R$ , where  $b \leq w_L \leq w_R \leq 1$ . That is,

$$q(w_L-) = q(w_L+), \quad q(w_R-) = q(w_R+), \quad q'(w_L-) = q'(w_L+), \quad q'(w_R-) = q'(w_R+). \quad (55)$$

The corresponding optimal investment strategy  $\pi(w)$  in these three regions are as follows:

1. If  $w$  is sufficiently high ( $w > w_R$ ),  $\pi(w)$  solves (28) with the property  $\pi(w) > 1$ .
2. If  $w$  is in the intermediate region ( $w_L \leq w \leq w_R$ ),  $\pi(w) = 1$ .
3. If  $w$  is sufficiently low ( $b \leq w < w_L$ ), the optimal investment strategy  $\pi(w)$  is given by

$$\pi(w) = \frac{\alpha}{\sigma^2\psi(w)} \leq 1, \quad (56)$$

where  $\psi(w)$  is the manager's effective risk aversion:

$$\psi(w) = -\frac{wq''(w)}{q'(w)}. \quad (57)$$

$\phi$	$\pi(1)$	$m(1)$	$n(1)$	$f(1)$	$e(1)$	$v(1)$
0	6.4515	0.2022	0.5346	0.7368	1	1.7368
0.25	5.4832	0.2312	0.4901	0.7213	1.1443	1.8656
0.5	4.9543	0.2437	0.4561	0.6998	1.2053	1.9051
0.75	4.6325	0.2497	0.4321	0.6818	1.2348	1.9166
1	4.4179	0.2527	0.4151	0.6678	1.2508	1.9186

Table 1: **The effects of managerial ownership on the investment strategy and various value functions under the same compensation contract ( $c = 2\%$ ,  $k = 20\%$ ) and no liquidation boundary ( $b = 0$ ).** The other parameter values are:  $\mu = 0.075$ ,  $r = 0.05$ ,  $\sigma = 0.107$ ,  $g = 0.05$ .

The cash flow to the manager (excluding incentive fees and prior to liquidation) is given by  $(c + \phi(\delta + \lambda))w$ . At the liquidation boundary  $b$ ,  $q(b) = \phi b$ . When calculating the manager's effective risk aversion  $\psi(w)$ , we use  $q(w)$  given in (57) as the value function.

We next use the case with no liquidation boundary ( $b = 0$ ) for illustration. We can also analyze the case with  $b = 0.5$  and obtain conceptually similar results. We proceed our analysis of ownership effects in two parts. First, we fix the managerial compensation contract, but vary managerial ownership. For each level of managerial ownership, we have different leverage and different values of investors' payoff. In equilibrium, capital markets are perfectly competitive and investors break even. Therefore, we need to adjust compensation contracts to ensure  $e(1) = 1$ . This two-part analysis allows us to decompose the total effects of managerial ownership into the *direct* and *equilibrium* effects.

**Part 1: The direct effect (holding the compensation contract at  $c = 2\%$ ,  $k = 20\%$ ).**

Table 1 reports the results for  $b = 0$  for various  $\phi$ . The first row repeats the results for the case with  $\phi = 0$ . Investors break even ( $e(1) = 1$ ) under 2-20 compensation and the manager collects all the surplus which is about 74 cents per dollar of AUM, i.e.  $f(1) = 0.7368$ . As we increase ownership  $\phi$ , leverage  $\pi(w)$  falls. The more skin the manager has in the game, the less the fund will be levered. The value of incentive fees  $n(w)$  thus falls, but the value of management fees  $m(w)$ , the value of investors' payoff  $e(w)$ , and the total value of the fund  $v(w)$  all increase. Quantitatively, the effects are significant. For example, the value of investors' payoff  $e(1)$  increases by 23.48 percent and the total fund value  $v(1)$  increases by 9.7 percent, when ownership  $\phi$  increases from zero to 50%. However, the surplus created from

better incentive alignments and lower leverage is unlikely to accrue to investors because capital markets are competitive ( $e(1) = 1$ ). We next analyze the *equilibrium* impact of managerial ownership  $\phi$  on leverage and various value functions by adjusting the managerial compensation contract to satisfy  $e(1) = 1$ .

**Part 2: The market equilibrium effect (adjusting  $(c, k)$  so that  $e(1) = 1$ ).** In general for any managerial ownership  $\phi$ , leverage depends on  $w$ . However, for an important special case, the manager's of excessive risk taking due to incentive fees is exactly balanced by his own equity exposure. Below, we summarize our model's results for this cutoff ownership  $\bar{\phi}$ , where the manager chooses a constant level of leverage which maximizes the fund's total value and is independent of  $w$ .

**Proposition 4** *Consider the setting with no liquidation boundary ( $b = 0$ ) but with the following managerial ownership:*

$$\phi = \bar{\phi} \equiv 1 - \frac{\alpha^*}{\delta + \lambda}, \quad (58)$$

where  $\alpha^*$  is given by (14). The optimal leverage is time-invariant, independent of  $w$ , and is given by (15) as in the benchmark model. The equilibrium compensation contract satisfying  $e(1) = 1$  requires setting the incentive fees as follows:

$$k = \frac{\delta + \lambda - (\alpha^* - c)}{\delta + \lambda - \eta(\alpha^* - c)} - 1, \quad (59)$$

where the parameter  $\eta > 1$  is given by

$$\eta = \frac{(\pi^* \sigma)^2 - 2(\alpha^* + r - g - c) + \sqrt{((\pi^* \sigma)^2 - 2(\alpha^* + r - g - c))^2 + 8(r + \lambda - g + \delta)(\pi^* \sigma)^2}}{2(\pi^* \sigma)^2}.$$

The various value functions are explicitly given by

$$n(w) = \frac{k}{\eta(k+1) - 1} w^\eta, \quad (60)$$

$$f(w) = \frac{c}{c + \delta + \lambda - \alpha^*} w + \frac{\delta + \lambda - \alpha^*}{c + \delta + \lambda - \alpha^*} \frac{k}{\eta(k+1) - 1} w^\eta, \quad (61)$$

$$e(w) = \frac{\delta + \lambda}{c + \delta + \lambda - \alpha^*} w - \frac{\delta + \lambda}{c + \delta + \lambda - \alpha^*} \frac{k}{\eta(k+1) - 1} w^\eta, \quad (62)$$

where  $k$  is given by (59). The value of management fees is given by  $m(w) = f(w) - n(w)$ . The total value for the manager including both fees and equity is  $q(w) = f(w) + \phi e(w) = w$ . Finally, the total fund's value is  $v(w) = e(w) + f(w)$ .



The time-invariant levered alpha (after fees) is given by (14). Within the set of contracts featuring incentive fees ( $k > 0$ ), the only level of managerial ownership that makes the manager choose the optimal time-invariant leverage  $\pi^*$  (15) is  $(1 - \alpha^*/(\delta + \lambda))$ , as given in (58). For all other levels of managerial ownership  $\phi$ , the manager either engages in more risk taking or behaves a bit more conservatively.

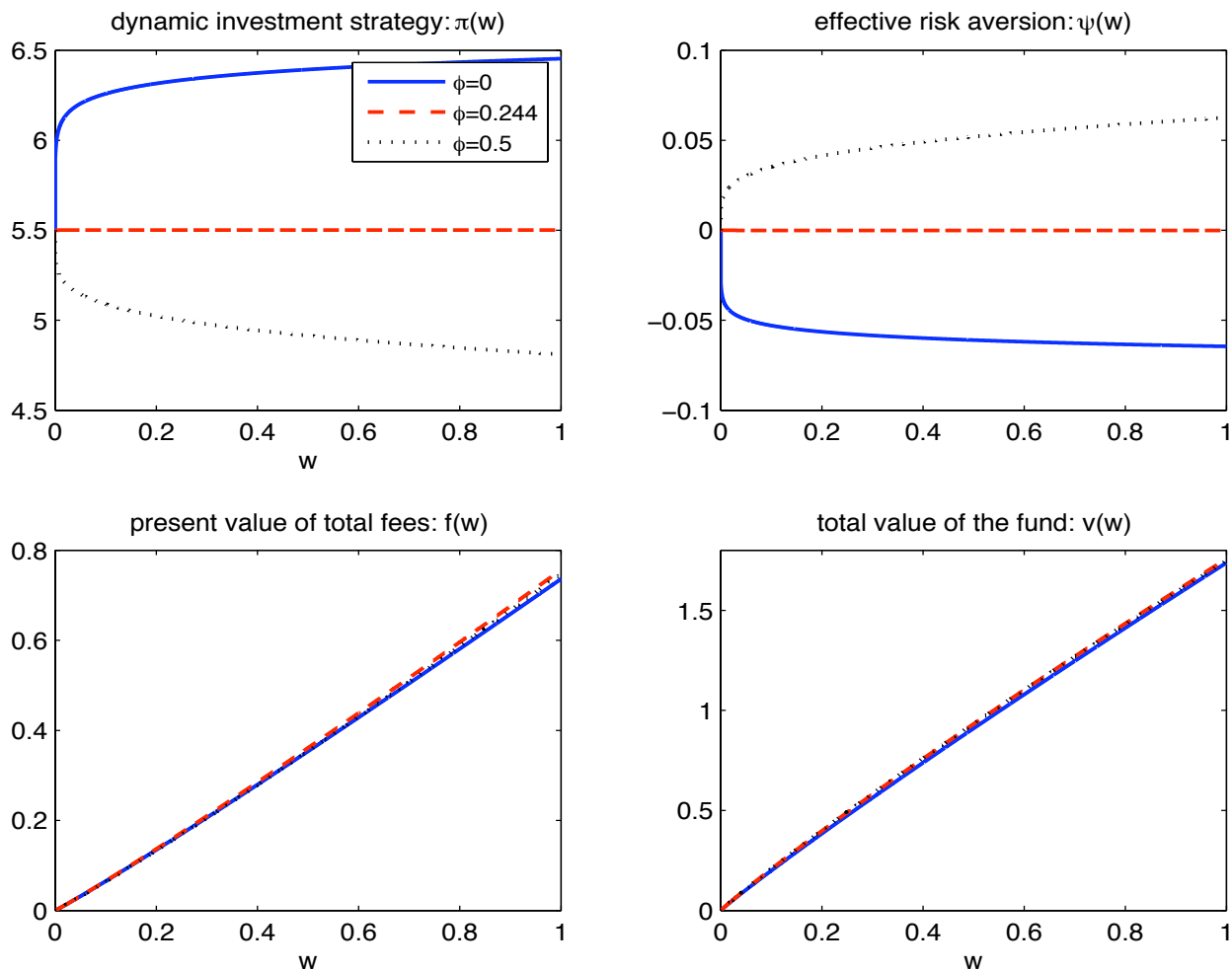


Figure 9: **Dynamic investment strategies  $\pi(w)$ , effective risk aversion  $\psi(w)$ , the manager's total value  $q(w)$  and present value of the fund for different managerial ownership  $\phi$  when the liquidation level is  $b = 0$ .** For all three cases, the managerial compensation contract  $(c, k)$  are respectively (2%, 20%), (2%, 27.4%), (2%, 33.2%) which satisfy  $e(1) = 1$ . Other parameter values are:  $r = 0.05$ ,  $\alpha = 0.025$ ,  $\sigma = 10.7\%$ ,  $\lambda = 0.05$ ,  $g = 0.05$ ,  $\delta = 0.05$ ,  $\theta_1 = 0.25\%$ , and  $\theta_2 = 0.5\%$ .

Figure 9 plots investment strategy  $\pi(w)$ , the manager's effective risk aversion  $\psi(w)$ , the total value of fees  $f(w)$ , and the total value of the fund  $v(w)$  under three different managerial compensation contracts. We choose three contracts with the following equilibrium pairings of  $(k, \phi)$ : (i)  $k = 20\%$  and  $\phi = 0$ , (ii)  $k = 27.4\%$  and  $\phi = 0.244$ , and (iii)  $k = 33.2\%$  and  $\phi = 0.5$ . In all three contracts,  $c$  is fixed at 2%. Importantly, investors break even in all three contracts:  $e(1) = 1$ .

First, consider the case with  $\phi = \bar{\phi} = 0.244$  and  $k = 27.4\%$ , which is covered by Proposition 4. We summarize the model's predictions below.

- The optimal leverage ratio is time invariant and is equal to  $\pi^* = 5.5$  as in Proposition 1 and the manager effectively behaves in a risk-neutral way (i.e.  $\psi(w) = 0$ ).
- Both the managerial value  $q(w)$  including both fees and *pro rata* share of investors' payoff, and the total value of the fund are linear:  $q(w) = w$  and  $v(w) = 1.756w$ , respectively.

If managerial ownership is lower than the cutoff level  $\bar{\phi} = 0.244$ , the incentive fee induced risk-taking motive dominates. The manager behaves in a risk seeking manner and therefore, leverage increases with  $w$ . On the other hand, if managerial ownership  $\phi$  is higher than the cutoff level,  $\bar{\phi} = 0.244$ , the ownership effect dominates. The manager behaves effectively in a risk-averse manner and leverage decreases with  $w$ . However, quantitatively, the valuation effect seems to be of second-order importance across these different ownership levels once we impose the equilibrium capital markets. The bottom two panels in Figure 9 confirm the results: the value of total fees  $f(w)$  and the total value of the fund  $v(w)$  are almost indistinguishable across these three different contracts.

## 8 The value of dynamically adjusting leverage

We now decompose the value of managerial rents and the total fund value into an unlevered component and the net increase in value due to leverage. First, we summarize the results in the benchmark case (without endogenous leverage decisions) as in GIR.

## 8.1 A pure valuation model ( $\pi(w) = 1$ ): GIR

GIR incorporates the effects of a high-water mark on the valuation of management and incentive fees without allowing for managerial leverage decisions. Their model can thus be viewed as a special case of ours with  $\pi(w)$  fixed at unity for all  $w$ . The following theorem summarizes the main results in GIR.

**Theorem 4** *Fixing  $\pi(w) = 1$  at all times and for all  $w$ , we have the following closed-form solutions for various value functions:*

$$n(w) = \frac{k(w^\eta - b^{\eta-\zeta}w^\zeta)}{\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1)}, \quad (63)$$

$$f(w) = \frac{c}{c + \delta + \lambda - \alpha}w + \frac{(\delta + \lambda - \alpha)k + (\zeta(1+k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1))}w^\eta - \frac{b^{\eta-\zeta}(\delta + \lambda - \alpha)k + (\eta(1+k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1))}w^\zeta, \quad (64)$$

$$e(w) = \frac{\delta + \lambda}{c + \delta + \lambda - \alpha}w - \frac{(\delta + \lambda)k + (\zeta(1+k) - 1)(c - \alpha)b^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1))}w^\eta + \frac{b^{\eta-\zeta}(\delta + \lambda)k + (\eta(1+k) - 1)(c - \alpha)b^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\eta(k+1) - 1 - b^{\eta-\zeta}(\zeta(1+k) - 1))}w^\zeta. \quad (65)$$

where  $\eta$  and  $\zeta$  are given by

$$\eta = \frac{\sigma^2 - 2(\mu - g - c) + \sqrt{(\sigma^2 - 2(\mu - g - c))^2 + 8(r + \lambda - g + \delta)\sigma^2}}{2\sigma^2} > 1 \quad (66)$$

and

$$\zeta = \frac{\sigma^2 - 2(\mu - g - c) - \sqrt{(\sigma^2 - 2(\mu - g - c))^2 + 8(r + \lambda - g + \delta)\sigma^2}}{2\sigma^2} < 0. \quad (67)$$

In addition, we have  $m(w) = f(w) - n(w)$  and  $v(w) = e(w) + f(w)$ .

Both incentive fees and the liquidation boundary ( $b > 0$ ) generate option features but for different reasons. Incentive fees encourage risk taking and hence tend to make the PV of fees  $f(w)$  convex in  $w$ . This effect is stronger for higher  $w$ . On the other hand, liquidation boundary induces the manager to behave conservatively because the manager does not want to lose future fees due to liquidation. This channel tends to make value function  $f(w)$  concave. This effect is stronger for lower  $w$ . Therefore,  $f(w)$  is convex for high enough  $w$  and concave for  $w$  sufficiently close to  $b$ .

The manager creates value in two ways: his skills (i.e. alpha) and the use of leverage. Using the GIR as the benchmark, we next quantify the value of leverage. Note that M&M

Ownership $\phi$	Optimal Leverage			NPV of Leverage	
	$\pi$	$f(1)$	$v(1)$	$\Delta f(1)$	$\Delta v(1)$
0	6.4515	0.7368	1.7368	0.4863	0.4870
0.25	5.4832	0.7213	1.8655	0.4708	0.6157
0.50	4.9543	0.6998	1.9045	0.4493	0.6547
0.75	4.6325	0.6818	1.9160	0.4313	0.6662
1	4.4179	0.6678	1.9180	0.4173	0.6682

Table 2: **The NPV of leverage for various levels of ownership.** The compensation contract is fixed at 2-20 ( $c = 2\%$ ,  $k = 20\%$ ) and there is no liquidation boundary ( $b = 0$ ). For the GIR benchmark case:  $f(1) = 0.2505$  and  $v(1) = 1.2498$ . The other parameter values are:  $\mu = 0.075$ ,  $r = 0.05$ ,  $\sigma = 0.107$ ,  $g = 0.05$ .

does not hold in our model because of managerial skills and financing costs. Leverage can magnify the value from managerial skills by increasing the gross size of the investment.

## 8.2 The net present value (NPV) of leverage

Consider the settings with  $b = 0$ . Without managerial ownership ( $\phi = 0$ ) and under 2-20 ( $c = 2\%$  and  $k = 20\%$ ), the manager chooses leverage to maximize the total value of fees yielding  $f(1) = 0.74$ . Because investors break even ( $e(1) = 1$ ), the total fund's value is thus  $v(1) = 1.74$ . However, without leverage ( $\pi = 1$ ) as in GIR, the value of total fees is substantially lower:  $f(1) = 0.25$ . The total value of the fund is also much lower:  $v(1) = 1.25$ . Leverage almost tripled the manager's value  $f(1)$  from 0.25 to 0.74. Intuitively, leverage allows the manager to capitalize on his skills (alpha).

Table 2 calculates the NPV of leverage for various levels of managerial ownership. We hold the compensation at 2-20 for all rows. Leverage decreases with managerial ownership due to incentive alignments. As a result, the value of total fees  $f(1)$  decreases with ownership, but the total fund's value  $v(1)$ , the sum of  $f(1)$  and  $e(1)$ , increases with  $\phi$ . Compared with the GIR benchmark with no leverage where  $f(1) = 0.25$  and  $v(1) = 1.25$ , the NPV of leverage for the value of fees,  $\Delta f(1)$ , remains significant, but decreases from 0.49 to 0.42 when  $\phi$  increases from 0 to 1. The NPV of leverage for the total fund's value,  $\Delta v(1)$ , increases from 0.49 to 0.67 when managerial ownership increases from 0 to 1.

Next, we quantify the effects of the investors' liquidation option on the NPV of leverage. As we increase the liquidation boundary  $b$  from 0 to 0.5, the manager's investment  $\pi$  becomes

Liquidation Boundary	Optimal Leverage			GIR Benchmark		NPV of Leverage	
	$b$	$\pi(1)$	$f(1)$	$v(1)$	$f(1)$	$v(1)$	$\Delta f(1)$
0	6.4515	0.7368	1.7368	0.2505	1.2498	0.4863	0.4870
0.25	5.3944	0.5318	1.6230	0.2503	1.2497	0.2815	0.3733
0.50	2.8287	0.2763	1.3055	0.2352	1.2313	0.0411	0.0742

Table 3: **The NPV of leverage with various liquidation boundaries.** The compensation contract is fixed at 2-20 ( $c = 2\%$ ,  $k = 20\%$ ) and there is no manager’s ownership. The other parameter values are:  $\mu = 0.075$ ,  $r = 0.05$ ,  $\sigma = 0.107$ ,  $g = 0.05$ .

increasingly conservative, decreasing from 6.45 to 2.83. As result, the value of total fees  $f(1)$  drops significantly from 0.74 to 0.28 when  $b$  increases from 0 to 0.5, and the total fund’s value  $v(1)$  decreases from 1.74 to 1.31. Note that similar patterns also hold for the GIR benchmark when investment is exogenously fixed with no leverage and no cash:  $\pi = 1$ . Quantitatively, the decreases of  $f(1)$  and  $v(1)$  are insignificant for the GIR benchmark. The value of fees  $f(1)$  and the total fund’s value decrease from 0.25 to 0.24 and from 1.25 to 1.23, respectively, as  $b$  increases from 0 to 0.5. The NPV of changing leverage for the manager  $\Delta f(1)$  thus drops significantly from 0.49 to 0.04 when liquidation boundary is increased from 0 to 0.5. Similarly, the NPV of changing leverage for the total fund value  $\Delta v(1)$  also decreases significantly, from 0.49 to 0.07. In our framework, imposing a liquidation boundary is inefficient because performances do not reveal anything about managerial skills and fund liquidation is thus never optimal in our model even following persistent disappointing performances.

## 9 Conclusions

Hedge fund managers are paid via both management fees and high-powered incentive fees. The latter is indexed to the stochastic and time-varying endogenous evolution of the high-water mark, which is effectively the running maximum of the AUM. While this compensation structure potentially rewards managerial talent, it also invites agency conflicts between the manager and investors. The fee structure gives the manager strong incentives to dynamically manage the fund’s leverage to maximize the present value of total fees, but not investors’ payoff. Under varying circumstances, management fees and incentive fees carry different relative weights towards total managerial rents. Investors rationally anticipate dynamically changing managerial incentives and price accordingly their investments in the fund.

The two key state variables in our model are the AUM and the HWM. In our homogeneous framework, the ratio between the AUM and the high-water mark determines the fund's dynamic leverage decisions. We show that high-powered incentives in the form of fees significantly encourage managerial risk taking. Leverage significantly increases the present value of managerial rents and the total fund value because leverage allows the manager to capitalize on their skills using a bigger pool of capital in a non-M&M world (with frictions). Granting investors options to liquidate the fund substantially curtails the manager's risk-taking incentives, effectively making the manager behave in a risk-averse manner. Having the manager co-invest in the fund as a limited partner also helps to align incentives and reduce agency conflicts.

For economic intuition and analytical tractability, we have intentionally chosen various modeling ingredients in order to keep the model setup homogenous in order to obtain analytical solutions. There are a few directions for future research. In our model, the manager's skill is known. However, in reality, investors may need to learn about the manager's skill. Extending the model to allow investors to learn about managerial skills is left for future research. Additionally, the recent crisis reveals that market liquidity and funding liquidity are first-order issues (Brunnermeier and Pedersen (2009)), and that funding costs can increase significantly in crisis periods. Anticipating these liquidity problems, the manager rationally chooses the fund's leverage and prudently manages risk in a state contingent way (Dai and Sundaresan (2010)). In future work, we plan to integrate market and funding illiquidity into our framework. Similarly, managerial skills (alpha) can also be time-varying because market investment opportunities are clearly time-varying. Finally, money chases performances and hence managerial rents depend on both the AUM/the performance of the current fund and in the future. Brown, Goetzmann, and Park (2001) point out that career concerns and incentives for collecting fees from new money flows may mitigate managerial risk-taking incentives. We leave these issues for future research.

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# Appendices

## A Technical details

**Proposition 1.** Without incentive fees, the manager chooses optimal investment  $\pi^*$  to maximize his value  $F^*(W)$  by solving the following HJB equation:

$$(r + \lambda)F^*(W) = \max_{\pi} cW + (\pi\alpha + r - \gamma(\pi) - \delta - c)WF_W^*(W) + \frac{1}{2}\pi^2\sigma^2W^2F_{WW}^*(W). \quad (\text{A.1})$$

With no incentive fees, we can conjecture that  $F^*(W) = xW$ , where  $x$  is a constant to be determined. The FOC for leverage is given by

$$\alpha = \gamma'(\pi^*), \quad (\text{A.2})$$

The above gives the optimal leverage provided that the solution indeed satisfies  $\pi^* > 1$ , which is implied by  $\gamma'(1) < \alpha$ . Otherwise,  $\pi^* = 1$ .

Substituting the optimal investment leverage into the HJB equation (A.1), we have

$$F^*(W) = M^*(W) = \frac{c}{c + \delta + \lambda - \alpha^*}W. \quad (\text{A.3})$$

Similarly, we obtain explicit value functions for  $E^*(W)$  and  $V^*(W)$  given in Proposition 1.

**Theorem 1.** Using the FOC for  $\pi$  in the HJB equation (24), we have

$$\pi = -\frac{(\alpha - \gamma'(\pi))F_W}{\sigma^2WF_{WW}}. \quad (\text{A.4})$$

We have also verified the second-order condition. We conjecture that the value function  $F(W, H)$  takes the following homogeneous form in  $W$  and  $H$ :

$$F(W, H) = f(w)H. \quad (\text{A.5})$$

The above conjecture implies the following results:  $F_W(W, H) = f'(w)$ ,  $F_{WW}(W, H) = f''(w)/H$ , and  $F_H(W, H) = f(w) - wf'(w)$ . Substituting (A.4) into the HJB equation (24) and boundary conditions (26), we obtain (27)-(31) in Theorem 1.

**Proposition 2.** Applying the Ito's formula to (4) and (5), we obtain the following dynamics for  $w$ :

$$\begin{aligned} dw(t) &= d\frac{W(t)}{H(t)} = \frac{dW(t)}{H(t)} - \frac{W(t)dH(t)}{H(t)^2} \\ &= \mu_w(w)dt + \pi(w(t))w(t)\sigma dB(t) - dJ(t), \end{aligned} \quad (\text{A.6})$$

where  $\mu_w(w)$  is given in (34). Applying the standard differential equation pricing methodology to the present values  $M(W, H)$  given in (6),  $N(W, H)$  given in (7) and  $E(W, H)$  given in (9), we obtain (35), (36), and (37) for  $m(w)$ ,  $n(w)$ , and  $e(w)$ , respectively.

Now consider the boundary behavior, i.e. when  $W(t) = H(t)$ . Using the same logic as the one for  $F(W, H)$ . Consider the scenario where the asset value increases by  $\Delta H$  over a small time interval, the high-water mark is reset to  $H + \Delta H$ . Because value functions are continuous, we thus have

$$M(H + \Delta H, H) = M(H + \Delta H - k\Delta H, H + \Delta H), \quad (\text{A.7})$$

$$N(H + \Delta H, H) = k\Delta H + N(H + \Delta H - k\Delta H, H + \Delta H), \quad (\text{A.8})$$

$$E(H + \Delta H, H) = E(H + \Delta H - k\Delta H, H + \Delta H). \quad (\text{A.9})$$

Therefore, we have:

$$kM_W = M_H, \quad kN_W = k + N_H, \quad kE_W = E_H. \quad (\text{A.10})$$

Similarly, using the homogeneous property, we may write  $M(W, H) = m(w)H$ ,  $N(W, H) = n(w)H$ , and  $E(W, H) = e(w)H$ . Finally,  $W = 0$  is an absorbing state and therefore  $M(0, H) = N(0, H) = E(0, H) = 0$ . Substituting these results into valuation equations, we obtain results in Proposition 2.

**Theorem 2, Proposition 3, and Theorem 3.** The proofs for Theorem 2, Proposition 3, and Theorem 3 are similar to the ones for Theorem 1 and Proposition 2. We have three regions for the optimal investment strategy  $\pi$  because the manager is now also short in the liquidation option. If the manager chooses no leverage and invests in the risk-free asset, then the financing cost is zero. Managerial concern for survival may make this strategy optimal for sufficiently low  $w$ . As a result, the optimal  $\pi$  can be less than unity, i.e.  $\pi < 1$  for  $w \leq w_L$ . Because the marginal cost of financing at  $\pi = 1+$  is strictly positive ( $\theta_1 > 0$ ), this

potentially makes the manager choose long-only with no risk-free asset (i.e.  $\pi = 1$ ) for a range of  $w$ , i.e.  $w \in (w_L, w_R)$ . Finally, we determine the cutoff points  $w_L$  and  $w_R$  by using the manager's optimality, which implies that the manager's value function is continuously differentiable at  $w_L$  and  $w_R$ .

**Proposition 4.** With no liquidation boundary ( $b = 0$ ) and managerial ownership  $\bar{\phi}$ , we show that the solution  $q(w)$  to the following ODE is linear in  $w$ :

$$(r + \lambda - g + \delta)q(w) = [c + \phi(\delta + \lambda)]w + \mu_w(w)q'(w). \quad (\text{A.11})$$

if and only if managerial ownership  $\phi$  is given by (58). Other results immediately follow. Finally, the equilibrium compensation contract which requires  $e(1) = 1$  implies (59).

**Theorem 4.** GIR's model is a special case of our model with investment constant strategy fixed at  $\pi(w) = 1$  for all  $w$ . Using our earlier results, we can calculate  $f(w)$  as follows:

$$(r + \lambda - g + \delta)f(w) = cw + (\mu - g - c)wf'(w) + \frac{\sigma^2 w^2 f''(w)}{2}. \quad (\text{A.12})$$

This is a standard linear second-order ODE with the following closed-form solution:

$$f(w) = xw + y_1 w^\eta + y_2 w^\zeta, \quad (\text{A.13})$$

where

$$x = \frac{c}{c + \delta + \lambda - \alpha}, \quad (\text{A.14})$$

$$\eta = \frac{\sigma^2 - 2(\mu - g - c) + \sqrt{(\sigma^2 - 2(\mu - g - c))^2 + 8(r + \lambda - g + \delta)\sigma^2}}{2\sigma^2}, \quad (\text{A.15})$$

$$\zeta = \frac{\sigma^2 - 2(\mu - g - c) - \sqrt{(\sigma^2 - 2(\mu - g - c))^2 + 8(r + \lambda - g + \delta)\sigma^2}}{2\sigma^2}. \quad (\text{A.16})$$

Note that  $\eta$  and  $\zeta$  are the positive and negative roots of the following (fundamental) quadratic equation in options analysis:

$$(r + \lambda - g + \delta) = (\mu - g - c)z + \frac{\sigma^2}{2}z(z - 1). \quad (\text{A.17})$$

To ensure sensible economics (i.e. positive and finite  $x$ ), we require  $\lambda + \delta + c - \alpha > 0$ , i.e. equivalently  $\eta > 1$ . Using the natural absorbing boundary condition  $f(b) = 0$  and

$f(1) = (k + 1)f'(1) - k$ , we obtain

$$y_1 = \frac{(\delta + \lambda - \alpha)k + (\zeta(1 + k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\eta(k + 1) - 1 - b^{\eta-\zeta}(\zeta(1 + k) - 1))}, \quad (\text{A.18})$$

$$y_2 = -\frac{b^{\eta-\zeta}(\delta + \lambda - \alpha)k + (\eta(1 + k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha)(\eta(k + 1) - 1 - b^{\eta-\zeta}(\zeta(1 + k) - 1))}. \quad (\text{A.19})$$

Closed-form expressions for other value functions can be similarly derived.